The Mathematics of Uncertainty

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To simplify presentation, summary definitions and results are provided. Publications available upon request

Rare Events with Important Consequences: Black Swans

Market Crashes, Natural Hazards: earthquakes, tsunamis, major episodes of extinction, catastrophic climate change

This research is about the foundations of probability when catastrophic events are at stake

It provides a new axiomatic foundation requiring sensitivity to the measurement of both rare and frequent events

The study culminates with a representation theorem proving existence and representing all probabilities satisfying *three axioms*

The last of the axioms requires sensitivity to rare events a property that is desirable but not satisfied by standard probabilities that neglect outliers and "fat tailed" distributions: these are the norm rather than the exception in empirical observations

We also show the connection between our axioms and the Axiom of Choice at the foundation of mathematics

Catastrophic Risks are Black Swans

Pentagon's recent report on Climate Change

A recent Pentagon report finds that *climate change* over the next 20 years could result in a global catastrophe costing millions of lives in wars and natural disasters. Potentially most important national security risk

http://www.guardian.co.uk/

environment/2004/feb/22/usnews.theobserver#att-most-commented

http://www.nytimes.com/

2009/08/09/science/earth/09climate.html?_r=2&pagewanted=1

http://wwfblogs.org/climate/

content/climate-change-climbs-ranks-pentagon-and-cia-0

Our research on the Foundations of Probability with Black Swans provides new foundations for probability, statistics and risk management that improve the measurement, management and mitigation of catastrophic risks.

It updates Mathematical and Economic tools for probability and optimal statistical decisions.

Mathematics of Uncertainty

Axioms for *relative likelihoods* or *subjective probability* were introduced more than half a century ago by Villegas, Savage, De Groot, others.

In parallel, Von Neumann and Morgenstern, Hernstein & Milnor, Arrow introduced axioms for *decisions making under uncertainty.*

The two theories are quite different. One focuses on *how things are*, the other on *how we make decisions*.

They are however paralell. Both provide classic tools for measuring and evaluating risks and taking decisions under uncertainty.

US Congress requires Cost Benefit Analysis of budgetary decisions.

Pentagon focus on extreme cases: security decisions that prevent the worst possible losses.

Classic theories neglect the measurements of extreme situations and rare events with important consequences, the type of catastrophic event that the Pentagon identifies in its recent report.

Evaluations of extreme events, and decisions to prevent and mitigate extreme losses contrast with standard statistical approaches and decisions that use "averages" - weighted by probabilities of occurence.

The purpose of this research is to correct this bias and update existing theory and practice to incorporate the measurement and management of catastrophic risks - focusing on average as well as extremal events.

Traditional Probabilities neglect rare events

Chichilnisky (2010, 2010a) show that traditional probability and statistics neglect rare events no matter how important their consequences. Based on Gaussian or 'normal' distributions (countably additive measures) they exclude 'outliers' and make 'fat tails' and power laws impossible. But these are the rule not the exception (e.g. finance, earth sciences)

Similarly, in decision making, *rationality* is often identified with

Expected Utility Optimization

$$\int_R u(c(t)d\mu(t)$$

For many years experimental and empirical evidence questioned the validity of the *expected utility* model.

Examples are the Allais Paradox, the Equity Premium Puzzle and the Risk Free Premium Puzzle in finance, and the new field of Behavioral Economics.

Discrepancies are most acute when ‘black swans’ or ‘catastrophic risks’ are involved.

This led to adopting "irrational" interpretations of human behavior that seem unwarranted and somewhat unproductive.

Catastrophic Risks are Black Swans

Savage (1963) defined a different foundation of statistics, where subjective probabilities are *finite additive measures*. Controversial, since his distributions give all weight to rare events. Examples Chichilnisky (2010, 2010a).

New Mathematics of Uncertainty: A middle ground

The new foundations of probability and statistics we lead to new distributions measuring rare events more realistically than classical statistics. These are new type of probabilities or distributions that are neither finitely additive as in Savage nor countably additive as in De Groot - **they have elements of both**

New Mathematical Developments

for Evaluation and Management of Catastrophic Risks

- Traditional Axioms neglect catastrophic events

- *New axioms* for decisions under uncertainty - (Chichilnisky, 1993, 1996, 2000, 2002, 2006, 2009, 2010, 2010a)

- New Axioms and results coincide with Von Neumann's in the absence of catastrophic events - otherwise they are quite different

- *A new representation theorem* identifies new types of probability distributions.

- New Decision theory combine standard approaches (which average risk) with distinct reaction to catastrophic or extremal risks (which do not)

- Convex combinations of ‘countably additive’ (*absolutely continuous*) and *‘purely finitely additive’ measures. Both can be absolutely continuous with respect to the Lebesgue measure*

- Example: Optimize expected returns while minimizing losses in a catastrophe

- A natural decision criterion - but it is *inconsistent* with expected utility and standard statistical theory.

- Finding new types of subjective probabilities that are consistent with experimental evidence, a combination of finite and countably additive measures

New Mathematics of Uncertainty

- New theory appears to agree with experimental and empirical evidence

- Extends classic theory to problems with catastrophic events

- Creates new Mathematical Results and Tools in Topology, Measure Theory, Functional Analysis, and Stochastic Processes called "Jump - Diffusion" processes.

- Change the way we practice and teach Statistics, Decisions under Uncertainty, Stochastic Processes, Bayesian Analysis Risk Managent and Financial Economics.

Summary of Publications & Applications

- Time: Infinite Horizons & Sustainable Development (1993, 1996, 2000)
- Uncertainty: Choices with Catastrophic Risks (2000, 2002)
- Econometrics: ‘NP Estimation in Hilbert Spaces: The Limits of Econometrics’ (2006, 2008)
- Neuroeconomic Theory: ‘Topology of Fear’ (2009)
- Experimental Results: Choices with Fear (2007 and 2009, with Olivier Chanel)
- Survival & Human Freedom: Godel & Axiom of Choice (2007 - 8)

- Green Economics: Climate Change: (2008)

- New Foundations for Statistics: "The Foundation of Probability with Black Swans" (2010) and "The Foundation of Statistics with Black Swans" (2010)

The Mathematics of Uncertainty

Uncertainty is represented by a family of sets or events (observables) $U = \{U_\alpha\}$ whose union describes the universe $U = \cup\{U_\alpha\}$. The relative frequency or probability of an event is a positive real number $W(U)$ that measures how likely it is to occur. Additivity means that $W(U_1 \cup U_2) = W(U_1) + W(U_2)$ when $U_1 \cap U_2 = \emptyset$, and $W(\emptyset) = 0$.

Example: A system is in one of several states described by $R =$ real numbers. For risk management to each state $s \in R$ there is an associated outcome, so that one has $x(s) \in R^N$, $N \geq 1$.

A description of probabilities across all states is called a *lottery* $x(s) : R \to R^N$. The space of all lotteries L is therefore a *function space* L, which could be an L_p space, $p = 1, ..., \infty$. Under uncertainty *one makes decisions by ranking lotteries in* L *and choosing the highest ranking lottery*

Von Neumann-Morgenstern's (NM) axioms provided a mathematical formalization of how to rank lotteries.

Optimization according to such a ranking is called *'expected utility maximization'* and defines classic decision making under uncertainty.

Classic *Expected Utility*

The main result from VNM axioms is a *representation theorem*:

Theorem: (VNM, Arrow, De Groot, Hernstein and Milnor) A ranking over lotteries which satisfies the VNM axioms admits a representation by an *integral operator* $W : L \rightarrow R$, which has as a 'kernel' a countably additive measure over the set of states, with an integrable density. This is called *expected utility*.

Expected Utility Maximization

The VNM representation theorem proves that the ranking of lotteries is given by a function

$$W : L \to R,$$

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

where the real line R is the state space, $x : R \to R^N$ is a "lottery", $u : R^N \to R^+$ is a (bounded) "utility function" describing the utility provided by the (certain) outcome of the lottery in each state s, $u(x)$, and where $d\mu(s)$ is a standard countably additive measure over states s in R.

Ranking Lotteries

Relative likelihoods rank events by how likely they are to occur. To choose among risky outcomes, we rank lotteries. An event lottery x is ranked above another y if and only if W assigns to x a larger real number:

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where W satisfies

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

The optimization of *expected utility* W is a widely used procedure for evaluating choices under uncertainty.

What are Catastrophic Risks?

A catastrophic risk is a small probability (or *rare*) event which can lead to major and widespread losses.

Classic methods do not work:

We have shown (1993, 1996, 2000, 2002) that using VNM criteria undervalues catastrophic risks and conflicts with the observed evidence of how humans evaluate such risks.

Problem with VNM Axioms

Mathematically the problem is that the measure μ which emerges from the VNM representation theorem is countably additive implying that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in (observable) sets of states whose probabilities are lower than some threshold level $\varepsilon > 0$ depending on x and y.

This means that expected utility maximization is insensitive to small probablity events, no matter how catastrophic these may be.

Problem with VNM Axioms

Expected utility is insensitive to rare events.

Definition 1: A ranking W is called Insensitive to Rare Events when for every $x, y \in L$

$$W(x) > W(y) \Leftrightarrow W(x') > W(y')$$

whenever the lotteries x' and y' are obtained by modifying arbitrarily x and y in any set of states $S \subset R$, with a small probability ε, formally $\forall(x, y)\ \exists\varepsilon > 0$, $\varepsilon = \varepsilon(x, y)$:

$$W(x) > W(y) \Leftrightarrow W(x') > W(y')$$

for all x', y' such that

$$x' = x \text{ and } y' = y \text{ a.e. on } A \subset R : \mu(A) < \varepsilon.$$

Similarly,

Definition 2: A ranking W is called *Insensitive to Frequent Events* when $\forall(x, y)\exists M > 0$, $M = M(x, y)$:

$$W(x) > W(y) \Leftrightarrow: W(x') > W(y')$$

for all x', y' such that

$$x' = x \text{ and } y' = y \text{ a.e. on } A \subset R : \mu(A) > M.$$

Proposition 1: Expected utility is Insensitive to Rare Events.

Proof of Proposition 1:

The space of lotteries is contained in L_∞, the space of essentially bounded measurable functions, an assumption that is satisfied when utilities are bounded as is normally accepted and required to avoid the St Petersburg's paradox (see Arrow, 1963). The expected utility criterion ranks lotteries $x(s)$ in $L = L_\infty(R)$ as follows

$$x(s) \succ y(s) \Leftrightarrow W(x) = \int_{s\in R} u(x(s))d\mu(s) > W(y) = \int_{s\in R} u$$

Now

$$\begin{aligned} W(x) &= \int_{s\in R} u(x(s))d\mu(s) > W(y) = \int_{s\in R} u(y(s))d\mu(s) \\ &\Leftrightarrow \exists\delta > 0 : \int_{s\in R} u(x(s))d\mu(s) > \int_{s\in R} u(y(s))d\mu(s) \end{aligned}$$

Let

$$\varepsilon = \varepsilon(x, y) = \delta/6K$$

where

$K = Sup_{x\in L, s\in R} \mid u(x(s)) \mid$; K exists since $x(s), y(s) \in L_\infty$

If

$x' = x$ and $y' = y$ a.e. on S^c and $\mu(S) < \varepsilon$ then

$$| \int_{s \in R} u(x(s)) d\mu(s) - \int_{s \in R} u(x'(s)) d\mu(s) |< 2K\mu(S) <$$

and

$$| \int_{s \in R} u(y(s)) d\mu(s) - \int_{s \in R} u(y'(s)) d\mu(s) |< 2K\mu(S) < \delta/3$$

Therefore since

$$W(x') - W(y') = [W(x') - W(x)] + [W(x) - W(y)] + [W(y) - W$$

and $| [W(x') - W(x)] |< \delta/\varepsilon$

$$| [W(x) - W(y)] |> \delta$$

$$| [W(y) - W(y')] |< \delta/\varepsilon$$

it follows that

$$W(x') - W(y') > \delta/3 > 0$$

Therefore

$$x \succ y \Longrightarrow \int_{s \in R} u(x'(s)) d\mu(s) > \int_{s \in R} u(y'(s)) d\mu(s) \Longrightarrow x' \succ$$

Reciprocally

$$x' \succ y' \Longrightarrow \int_{s \in R} u(x(s))d\mu(s) - \int_{s \in R} u(y(s))d\mu(s) \Longrightarrow x \succ y$$

so that $\forall x, y \exists\ \varepsilon = \delta/6K$, where $K = K(x, y)$,

, $x \succ y\ \exists \varepsilon = \varepsilon(x, y) > 0 : x' \succ y'$ when $x = x'$ and $y = y'$ a.

Therefore by definition the expected utilility criterion is insensitive to small probability events.■

It follows that as defined by VNM, expected utility W is ill suited for evaluating catastrophic risks. The new axioms resolve this issue by requiring sensitivity to rare events.

New Axioms

Axiom 1: The ranking W is continous and linear (this is in common with VNM)

Axiom 2. The ranking W is sensitive to frequent events (this is common with VNM)

Axiom 3: The ranking W is sensitive to rare events

(this is new, and is the logical negation of Arrow's "Monotone Continuity" axiom (1963) and of De Groot's axiom SP_4)

Axioms 1 and 2 are standard, they are satisfied for example by expected utility

Axiom 3 is different and is not satisfied by expected utility as was shown in Proposition 1.

Old and new Mathematics of Uncertainty:

Topology holds the Key

Mathematically, VNM axioms postulate nearby responses to nearby events, or "continuity"; in the case of Arrow (1963) the requirement is actually called "Monotone Continuity" and in De Groot it is called Axiom SP_4. The objective in all cases is to give rise to a countably additive measure appearing as the "kernel" of the Expected Utility.

Nearby is measured by *averaging* distances.

However with catatrophic risks, we measure distances by *extremals*.

Mathematically the difference is as follows:

Average distance - the L_p norm ($p < \infty$)

$$\| f - g \|_p = (\int | (f-g)^p | dt)^{1/p}$$

and H^p norms in Sobolev spaces.

Extremal distance - the sup. norm of L_∞ :

$$\| f - g \|_\infty = ess \sup_R | (f-g) |$$

Our reaction to extremal catastrophic risks is extremal

Changing the topology, namely the way we measure distances, changes our approach to risk.

It leads to new ways to evaluate risk

The Mathematics of Uncertainty:

It leads to *Gaussian* measures combined with *singular* measures

Updating Von Neumann Axioms for Choice Under Uncertainty

Axiom 1: Sensitivity to Rare Events

Axiom 2: Sensitivity to Frequent Events

Axiom 3: Linearity and Continuity (in L_∞)

Similarly the axioms for probability with Black Swans are

Axiom 1: Probabilities are continuous and additive

Axiom 2: Probabilities are unbiased against rare events

Axiom 3: Probabilities are unbiased against frequent events

Axiom 1 *negates* Arrow's "Axiom of Monotone Continuity", an axiom that Arrow (1963) introduced to obtain Expected Utility. Indeed:

Theorem 1: The Monotone Continuity Axiom (Arrow, Milnor) is equivalent to "Insensitivity to Rare Events".

Our Axiom 3 is its logical negation.

Proof: See Theorem 2, "The Topology of Fear", JME, 2009

A Representation Theorem

Like VNM axioms, the new axioms lead to a (new) representation theorem.

Theorem 2 (Chichilnisky 1992, 1996, 2000, 2002)

There exist criteria or functionals $\Psi : L_\infty \to R$ that evaluate lotteries satisfying all three new axioms. All such functionals are defined by a convex combination of purely and countably additive measures, with both parts non zero.

Formally, there exists ν, $0 < \nu < 1$, a utility function $u(x) : R \to R$ and a countably additive regular measure μ on R, represented by an L_1 density Γ, such that the ranking of lotteries $\Psi : L_\infty \to R$ is of the form

$$\Psi(x) = \nu \int u(x(s))\Gamma(s)d\mu(s) + (1 - \nu)\Phi(u(x(s))).$$

where Φ denotes a purely finite measure on R.

Proof: The space of continuous linear real valued functions on the space $L_\infty(R)$ is called the "dual" of $L_\infty(R)$ and is denoted $L^*_\infty(R)$; this dual space is well known to consist of measures that are *countably additive* and also of *purely finitely additive* measures: A *countably additive measure* μ on R satisfies $\mu(\cup_{i=1,2,...}U_i) = \sum_{i=1,...\infty} \mu(U_i)$ whenever the measurable family of sets $\{U_i\}i = 1, 2, ...$ consists of pairwise disjoint sets, and if this property is not satisfied, the measure μ is still additive (by definition of a measure) but is not countably additive, and therefore μ is called a "purely finitely additive measure." We have shown in Proposition 1 that whenever the measure μ is countably additive, Axiom 3 is violated, since the function W is then a standard expected utility with a countably additive kernel. It is immediate to show that Axiom 2 is violated when μ is purely finitely additive. The only possibility left is that μ be a sum of a countably additive and a purely finitely additive measure, which is neither purely finitely additive nor countably additive when both summands are non zero. See also the proof in Chichilnisky (1996).

From the above Theorem it follows that when the sample contains no catastrophic events, the second axiom is void. Therefore the second component of Ψ "collapses" to its first component, and we have

Theorem 3:

In the absence of catastrophic events, the above functional Ψ agrees with VNM's Expected Utility criterion for evaluating lotteries.

New Results

Choices under Uncertainty with Finite States

Theorem 4: A convex combination of Expected Utility and the Maximin criterion satisfies the axioms proposed here.

Proof: Chichilnisky, 2007 see also Arrow and Hurwicz.

New Result on Limits of Econometrics

Non Parametric Estimation in Hilbert Spaces

with sample space R

Theorem 5: Insensitivy to Rare Events is equivalent to the statistical Assumption SP_4 in Degroot, comparing the relative likelihood of bounded and unbounded events. Both are Necessary and Sufficient for *NP* estimation in Hilbert Spaces on the sample space R.

New Result on Transition to Green Economics (2008)

Renewable Resource Optimization - Survival & Extinction

- Choice with the new Axioms is *equivalent to* optimizing expected utility plus a survival contraint on extinction

- The factor λ that links countable and finite measures, can be identified with the marginal utility of the renewable resource at the point of extinction.

The results extend to how to observe or measure events under uncertainty

- "The Foundations of Probability with Black Swans" - new types of probability distributions (2010)

- "The Foundations of Statistics with Black Swans" - new statistical treatments that are sensitive to catastrophic risks

Examples of the new criteria

Finance:

Maximize expected returns while minimizing the drop in a portfolio's value in case of a market downturn

Network optimization:

Electric grids: Maximize expected electricity throughput in the grid, while minimizing the probability of a "black out"

Stochastic Systems:

Jump - Diffussion Processes (Merton, 1985).

Deep Mathematical Roots

The construction of functions to repersent purely finitely additive measures is equivalent to Hahn Banach's theorem and therefore requires the Axiom of Choice.

Thus extreme responses to risk conjure up the 'Axiom of Choice' and create new types of probability distributions that are both countably and purely finitely additive measures, which have never been used before.

Surprisingly, the *sup norm topology* was already used by Gerard Debreu in 1953, to prove Adam Smith's *Invisible Hand Theorem*.

The practical implications of Debreu's results were not understood before. Yet Debreu published his 1953 results in the Proceedings of the US National Academy of Sciences – in an article introduced by Von Neumann himself.

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