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Abstract Market objectives can conflict with long-term goals. Behind the conflict is the *impatience* axiom introduced by T. Koopmans to describe choices over time. The conflict is resolved here by introducing a new concept, *sustainable markets*. These differ from Arrow-Debreu markets in that traders have *sustainable preferences* and no bounds on short sales. Sustainable preferences are sensitive to the basic needs of the present without sacrificing the needs of future generations and embody the essence of sustainable development (Chichilnisky in Soc Choice Welf 13(2):231–257, 1996a; Res

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Energy Econ 73(4):467–491, 1996b). Theorems 1 and 2 show that *limited arbitrage* is a necessary and sufficient condition describing diversity and ensuring the existence of a sustainable market equilibrium where the invisible hand delivers sustainable as well as efficient solutions (Chichilnisky in Econ Theory 95:79–108, 1995; Chichilnisky and Heal in Econ Theory 12:163–176, 1998). In sustainable markets prices have a new role: they reflect both the value of instantaneous consumption and the value of the long-run future. The latter are connected to the independence of the axiom of choice at the foundations of mathematics (Godel 1940).

Keywords Sustainability · Impatience · Axioms for sustainable preferences · Arbitrage · General equilibrium with infinite horizons

JEL Classification D4 · Q2 · Q5 · Z0

1 Introduction

Sustainable development requires satisfying the basic needs of the present without sacrificing the needs of future generations. This seems to set up a confrontation between market objectives, which are typically short term, and the requirements of sustainable development. Behind this confrontation is a standard feature of classic markets: the concept of *impatience*, an axiom that was introduced in Koopmans's seminal work on economics over time and is a requirement of the Arrow-Debreu theory of markets. A dollar tomorrow is worth less than a dollar today and this, in its simplest form, introduces an economic bias against the future (Chichilnisky 1996a,b; Chichilnisky and Heal 1998). The problem is quite general. Even in markets with infinite horizons, the existence of a solutions seems to require some form of *impatience*. For example, the "cone condition" of Chichilnisky and Kalman (1980), also known as "properness" and frequently to proved existence of market equilibrium with infinite horizons, still requires a form of impatience, see e.g., Yannelis and Zame (1986), and Chichilnisky (1993).[1]

Markets and sustainability seem opposed to each other. Is this a correct view of markets? A natural question is whether a society committed to sustainability must discard markets. This would be a major change, since markets are a widespread form of organization.

The article argues that it is possible to overcome a market's bias against the future. This requires defining new types of markets where traders have 'sustainable preferences', as introduced in Chichilnisky (1996a,b), and a corresponding definition of market equilibrium. Because markets follow the priorities of the traders, when traders have sustainable preferences markets become sustainable. This, in a nutshell, explains the results of the article.

Whether markets clash with sustainability hinges therefore on whether traders have *sustainable preferences*. These are a new type of preferences that overcome the impatience axiom: they are based on new axioms that require equal treatment of the present

[1] Chichilnisky (1993) showed that the original "cone condition" is the same as the later condition named "properness".

and the future, and were introduced by the author in Chichilnisky (1996a,b) see also comments in Lauwers (1993). These new axioms reflect an increasing body of empirical evidence about how humans value the long-term future (Chichilnisky 1996a,b). Based on the concept of sustainable preferences, this article defines *sustainable markets* that differ from standard Arrow-Debreu markets in two ways: traders (i) have *sustainable preferences* and (ii) can engage in short trades. Markets with sustainable preferences overcome impatience because they are neither dictatorial for the present nor for the future; they are in fact sensitive to the needs of both as required for sustainable development. In sustainable markets, market prices take a new role; they represent the value of instantaneous consumption as well as the value of future consumption. This approach resolves the conflict between a market's short-term objectives and the goals of sustainability, without eliminating market organization. We require instead a different type of market than the one we had until now, namely *sustainable markets*, which are not based on impatience but on sustainable preferences. Sustainable preferences are linear as time preferences, the same property satisfied by standard present discounted utility functions.

To reconcile the opposing needs of the present and the future, we establish the existence of equilibrium in sustainable markets. We show that when markets have sustainable preferences, a single condition—*limited arbitrage*—is necessary and sufficient for the existence of Pareto efficient market equilibrium, without bounds on short sales. This ensures the logical consistency of *sustainable markets.* To achieve this, we have to overcome three interlinked technical issues (i) continuity of preferences and prices, (ii) compactness of trading sets and efficient allocations, both of which (i) and (ii) are used to prove existence of solutions, and (iii) appropriate supporting prices for efficient allocations. With infinite horizons, compactness requires weaker topologies that can imply a form of impatience (Yannelis and Zame 1986). Furthermore supporting prices that are continuous with the sup norm can lead to paradoxical results (Chichilnisky and Kalman 1980). To resolve the continuity-compactness dilemma and avoid paradoxes we rely on the properties of sustainable preferences that are sensitive to the present and the future, and use the notion of "limited arbitrage" introduced in Chichilnisky (1991, 1995, 1994a, 1996c, 1998). Taken together, sustainable preferences and the notion of limited arbitrage, overcome the problems of impatience and unlimited short sales by limiting somewhat the diversity of the traders (Chichilnisky 1994b), bounding the "gains from trade" that they can achieve trading with each other, while creating sensitivity to the present and the future and ensuring existence of solutions. In Proposition 2, we show that limited arbitrage is equivalent to bounded gains from trade, and it ensures the compactness of Pareto utility allocations, as needed for the existence of solutions, based on earlier results by Chichilnisky (1991, 1995, 1998) and Chichilnisky and Heal (1984, 1998). To complete the existence result, Theorem 1 proves that, in a sustainable market, limited arbitrage is equivalent to the compactness of the set of Pareto efficient allocations, and Theorem 2 establishes that it is necessary and sufficient for the existence of sustainable as well as efficient competitive market equilibrium. Section 4 discusses the role of prices in a sustainable market economy. These assign economic value both to instantaneous and long-run consumption, providing a connection with the axiom of choice that is at the foundation of mathematics. A sustainable market equilibrium is shown to exist, and in a sustainable markets market prices take a new role: they represent the value of instantaneous consumption as well as the value of long-run future consumption. Indeed, when sustainable constrains are taken into consideration, new concepts and features of economics arise as demonstrated throughout this entire special volume on the economics of the global environment, see Asheim et al., Burniaux and Martins, Chipman and Tian (2011), Dutta and Radner (2011), Figuieres and Tidball (2011), Karp and Zhang (2011), Lauwers (2011), Lecocq and Hourcade (2011), Ostrom (2011) and Rezai et al. (2011).

2 Sustainable markets

This section defines *sustainable markets*.

2.1 Definitions

A competitive market has $H \geq 2$ traders and $N \geq 2$ commodities that are traded over time $t \in R_+$. The consumption of commodities yields utility $u(x(t))$ at each period of time t, where $x(t) \in R^N$, and $u(x) : R^N \to R_+$ is a concave increasing real valued function that represents instantaneous utility in period t. Following the classic work of Debreu (1953) and Chichilnisky (1996a,b, 2009a,b), one can view consumption paths over time $(x(t)_{t \in R^+})$ as elements of $L_\infty(R^N)$. Similarly, utility paths over time $f(t) = u(x(t))$ are elements of the linear space $L_\infty(R)$, where L_∞ is the space of all essentially bounded measurable real valued functions on R with the sup *norm* $\|f\| = ess \sup_{t \in R} |f(t)|$. In this context, a *preference over time* $U : L_\infty \to R$ is a real valued *linear* function ranking utility paths $u(x(t))$, while $U : L_\infty\left(R^N\right) \to R$ denotes the ranking of consumption paths $(x(t))$, which is based on a concave instantaneous utility $u : R^N \to R$, and is generally *non linear*. We say that the preference over time U is a *dictatorship of the present* when it disregards all utility beyond a period T, namely $U(f) > U(g) \Leftrightarrow U(f') > U(g')$ for any a.e. modification of f and g that occurs beyond T, i.e., when $f'(t) = f(t)$ and $g'(t) = g(t)$ a.e. for all $t > T$. A ranking is a *dictatorship of the future* when it disregards utility modifications in the present; formally, for any two paths f, g, there exists a period $T \in R : U(f) > U(g) \Leftrightarrow U(f') > U(g')$ for any a.e. modification of f, g that occurs prior to period T, i e. whenever $f'(t) = f(t)$ and $g'(t) = g(t)$ for all $t < T$. The logical negation of these two dictatorship properties defines *non-dictatorship* of the present and *non-dictatorship* of the future.

2.2 Axioms for sustainable preferences

A *sustainable preference U* is an increasing ranking that as a time preference satisfies three axioms.[2]

[2] The axioms for sustainable preferences were introduced in Chichilnisky (1996a,b), and similar axioms were introduced for the foundations of preferences under uncertainty, for NP econometrics (Chichilnisky 2009a,b), for relative likelihoods and the foundations of probability and statistics (Chichilnisky 2010a,b).

Axiom 1 $U : L_\infty \to R$ is continuous and linear[3]

Axiom 2 $U : L_\infty \to R$ is a non-dictatorship of the future

Axiom 3 $U : L_\infty \to R$ is a non-dictatorship of the present

These axioms were introduced in Chichilnisky (1996a,b). The first two are consistent with T. Koopman's classic axioms of choice over time and are satisfied by present discounted utility

$$U(f) = U(x(t)) = \int_{R^+} u(x(t))e^{-\delta t}\mathrm{d}t, \quad \delta > 1$$

where $f \in L_\infty$ represents a time path $u(x(t))$, δ is a time 'discount factor'. Observe that the present discounted utility $U(f)$ defined above is linear on utility paths $u(t)$, and thus satisfies Axiom 1, but it may not be linear in consumption x. Sustainable preferences that satisfy Axioms 1,2 and 3 are also linear on utility paths but may not be linear on consumption. The third axiom however is not satisfied by present discounted utilities (Chichilnisky 1996a,b). **Sustainable preferences** have been characterized in a representation theorem established in Chichilnisky (1996a,b, 2009a,b, 2010a,b) to be of the form

$$U(f) = \lambda U_1(f) + (1 - \lambda)U_2(f) \tag{1}$$

where $U_1(.)$ is a function in L_1 and $U_2(.)$ is in $L^*_\infty - L_1$[4], $0 < \lambda < 1$, both $U_1(f)$ and $U_2(f)$ are increasing and non-zero, and specifically:

$$U(f) = U(x(t)) = \lambda \int_{R^+} u(x)\phi(x)\mathrm{d}t + (1 - \lambda)\chi(u(x))$$

where $U_1(f) = \lambda \int_{R_+} u(x)\phi(x)\mathrm{d}t$, $U_2(f) = (1 - \lambda)\chi(u(x)$, $0 < \lambda < 1$, $\phi \in L_1$, e.g., $\phi(t) = \mathrm{e}^{-\delta t}$, and $\chi \in L^*_\infty - L_1$ is a purely finitely additive measure on R (for a proof see Chichilnisky (1996a,b, 2009a,b, 2010a,b).

2.3 Sustainable markets

Definition A sustainable market is an Arrow-Debreu market where traders have infinite horizons, no bounds on short sales and sustainable preferences over time.

A sustainable market economy can be represented as $E = \{X, \Omega_h, U_h : X \to R, h = 1, \ldots, H\}$. It has $H \geq 2$ traders indexed by $h = 1, 2, \ldots, H$, $N \geq 2$ commodities

[3] The time preference U ranks paths over time $u(t) \in L_\infty$ and Axiom 1 requires U to be continuous and linear. Observe that since the instantaneous utility function $u : R^N \to R$ is concave and need not be linear, the ranking of consumption paths need not be linear as a function of consumption, $x(t)$.

[4] L^*_∞ is the dual space of L_∞, the space of all continuous, linear, real valued functions on L_∞.

that are traded over time $t \in R_+$; the *consumption space* or *trading space* is the Banach space $X = L_\infty$ with the sup norm $\| \ . \ \|_{\sup}$ (Debreu 1953; Chichilnisky 1996a,b, 2009a,b); this assumption implies no bounds on short sales. $\Omega_h \in X$ represents trader $h's$ property rights, $\Omega = \sum_h \Omega_h$ represents society's total resources over time; and traders' preferences over time $U_h : L_\infty \to R_+$ are based as above on concave instantaneous utility $u_h : R^N \to R_+$ and define sustainable time preferences.

Traders may have zero endowments of some goods, and endowments could be negative or positive; since the trading space is $X = L_\infty$, short selling is allowed. We consider general preferences where the normalized gradients to indifference surfaces define either an open or a closed map on every indifference surface, namely (i) indifference surfaces contain no half-lines, for example strictly convex preferences, or (ii) the normalized gradients to any closed set of indifferent vectors define a closed set, for example linear preferences. In this article for simplicy, we identify case (ii) with linear preferences. The assumptions and the results of the paper are ordinal, and $U_h(0) = 0$ and $\sup_{x \in X} U_h(x) = \infty$. Preferences are increasing so that $U_h(x(t)) > U_h(y(t))$ when for all $t, x(t) \geq y(t)$ and for a set of positive Lebesgue measure, $x(t) > y(t)$. In addition, we assume the traders's preferences are uniformly non-satiated, which means that they can be represented by a utility U with a bounded rate of increase: for smooth preferences, which are Frechet differentiable, $\exists \varepsilon, K > 0 : \forall x \in X, K > \| DU(x) \| > \varepsilon$. If a utility function is uniformly non-satiated, its indifference surfaces are within a uniform distance from each other: $\forall r, s \in R, \exists N(r,s) \in R$ such that $f \in U^{-1}(r) \Rightarrow \exists y \in U^{-1}(s)$ with $\| f - g \| \leq N(r,s)$, see Chichilnisky and Heal (1998).

Assumption 1 Each trader has a sustainable time preference, satisfying Axioms 1,2 and 3, which is represented by an increasing, uniformly non-satiated function of consumption paths over time $U : L_\infty \to R^+$ based on a concave instantaneous utility $u : R^N \to R$ such that $U(0) = 0$ and $\sup_{f \in X} U(f) = \infty$.

Prices are real-valued linear functions on X that are continuous with the sup norm (Debreu 1953). The space of *feasible allocations* over time is $\{(f_1(t), \ldots, f_H(t)) \in X^H : \sum_{h=1}^H f_h(t) = \sum_{h=1}^H \Omega_h = \Omega\}$ To simplify notation when it is clear, we obviate the time variable t. A utility vector $U = (U_1(f_1), \ldots, U_H(f_H))$ is *feasible* if the allocation $(f_1, \ldots, f_H)$ is feasible.

The set of *individually rational feasible allocations* is the set of utility allocations $\{U_1(f_1), \ldots, U_H(f_H)\}$ that are feasible and preferred to the initial endowments, $\forall h, U_h(f_h) \geq U_h(\Omega_h)$. A utility vector $U = (U_1(f_1), \ldots, U_H(f_H))$—which need not be feasible—is *efficient or undominated* if there is no allocation $G = (g_1, \ldots, g_H)$ such that $\forall h, U_h(g_h) \geq U_h(f_h)$ and $U_k(g_k) > U_k(f_k)$ for some k, and there exists a sequence of feasible allocations $(f_1^j, \ldots, f_H^j)_{j=1,2,\ldots}$ such that $G = \lim_{j \to \infty}(f_1^j, \ldots, f_H^j)_{j=1,2,\ldots}$. A *feasible efficient allocation* is a feasible allocation that is also efficient.

The *Pareto Frontier* $P(E) \subset R_+^H$ is the set of individually rational and efficient feasible utility vectors. A *competitive equilibrium of the economy E* consists of a price vector $p^* \in X_+^*$ and an allocation $(f_1^*, \ldots, f_H^*) \in X^H$ such that f_h^* optimizes U_h over the budget set $B_h(p^*) = \{f \in X : \langle p^*, f \rangle = \langle p^*, \Omega_h \rangle\}$ and clears the markets

$\sum_{h=1}^{H} f_h^* - \Omega_h = 0$. A feasible allocation $(f_1, \ldots, f_H)$ is a *quasiequilibrium* when there is a price $p \neq 0$ with $\forall h, \langle p, \Omega_h \rangle = \langle p, f_h \rangle$, and $\langle p, g \rangle \geq \langle p, f_h \rangle$ for any g implies $U_h(g) \geq U_h(f_h)$. A quasi-equilibrium is a *competitive equilibrium* when $U_h(g) > U_h(f_h) \Rightarrow \langle p, g \rangle > \langle p, f_h \rangle$.

The following concept of a **global cone** contains global information about a trader since, in ordinal terms, the sequences in this cone achieve utility values that eventually exceed those of all trades. The global cone was introduced in Chichilnisky (1991, 1995, 1994a,b, 1996c,d), see also Chichilnisky and Heal (1998):

Definition The cone A_h consists of all sequences of net trades $\{f^j\}$ in X along which the hth trader's utility increases and exceeds that of any other vector in the space; it can be based on rays of directions in X along which the hth trader's utility exceeds eventually all utility values:

$$A_h(\Omega_h) = \left\{ \{f^j\} : \forall g \in X, \exists\, j : U_h(f^j) > U_h(g) \right\}$$

Definition The *global cone* $G_h(\Omega_h)$ is the set of all sequences of net trades in X along which the hth trader's utility never ceases to increase; it can be based on rays of directions with ever increasing utility:

$$G_h(\Omega_h) = \left\{ \{f^j\} :\sim \exists Max_j U_h(f^j) \right\}.$$

We assume that $G_h(\Omega_h)$ has a simple structure, which was established in different forms in Chichilnisky (1991, 1995, 1994b, 1998); Chichilnisky and Heal (1998): when preferences have no half-lines in their indifferences, case (i), then $G_h(\Omega_h)$ is the closure of $A_h(\Omega_h)$ and in case (ii) when preferences have half-lines in their indifference surfaces, for example linear preferences, then $G_h(\Omega_h) = A_h(\Omega_h)$.

Definition The *Market Cone* $D_h(\Omega_h)$ is

$$D_h(\Omega_h) = \left\{ p \in X : \forall \{g\} \in G_h(\Omega_h), \exists i : \langle g^j, p \rangle > 0 \text{ for } j > i \right\}$$

This is the set of all prices assigning eventually strictly positive value to net trades in the global cone. We assume the results of the following proposition, which was established in different forms in Chichilnisky (1991, 1995, 1994a,b, 1996c,d, 1998); Chichilnisky and Heal (1998), in proving the connection between limited arbitrage and the existence of a sustainable market equilibrium:

Proposition 1 *If a utility $U : X \to R$ is uniformly non-satiated, then*

(A) *$A(\Omega) \neq \emptyset$, and the cones $G(\Omega)$ and $D(\Omega)$ are all convex and uniform across vectors Ω in X.*[5] *For general preferences $G(\Omega)$ and $D(\Omega)$ may not be uniform,* Chichilnisky (1998); Chichilnisky and Heal (1998).

(B) *In case (i), preferences have no half-lines in their indifferences, $G_h = \overline{A_h}$; with linear preferences case $(ii) G_h = A_h$.*

[5] The cones $C(\Omega) = \{\{f\} \subset X : \lim_{j \to \infty} f^j = U(j^{j_0})$ for some $j_0\}$ are also convex and uniform across vectors Ω.

2.4 Limited arbitrage and gains from trade with short sales

This section defines limited arbitrage and provides an intuitive interpretation in terms of gains from trade. The following definitions and results are used in establishing the existence of a competitive equilibrium and are based on Chichilnisky (1991, 1995, 1994a,b, 1996c,d, 1998) and Chichilnisky and Heal (1998).

Definition *Gains from trade* are defined as

$$\mathcal{G}(E) = \sup \left\{ \sum_{h=1}^{H} (U_h(f_h) - U_h(\Omega_h)) \right\} \quad \text{where } \forall h,\ f_h = \left(f_h^j\right) \text{ satisfies}$$

$$\sum_{h=1}^{H} (f_h - \Omega_h) = 0 \quad \text{and} \quad U_h\left(f_h^{j+1}\right) > U_h\left(f_h^j\right) > U_h(\Omega_h) \geq 0.$$

Definition The economy E satisfies *limited arbitrage* when

$$\bigcap_{h=1}^{H} D_h \neq \emptyset \tag{2}$$

Geometrically, *Limited Arbitrage* (2) bounds arbitrage opportunities in the economy by limiting the utility that can be achieved by the traders when trading with each other. Under the assumptions, proposition 2 below applies in case (i) and (ii): either indifference surfaces contain no half-lines (e.g., strictly convex preferences) of (ii) linear preferences.

Proposition 2 *Limited arbitrage implies bounded gains from trade, namely* $\mathcal{G}(E) < \infty$.

Proof The proof relies on limited arbitrage and follows the proofs of similar propositions in Chichilnisky (1991, 1995, 1998), Chichilnisky and Heal (1998) adapted to markets with sustainable preferences. Along the way, we also highlight properties of sustainable preferences that are useful for understanding the structure of sustainable preferences and the existence of a competitive equilibrium in sustainable markets.

Assume E has limited arbitrage and without loss of generality that $\forall h, \Omega_h = 0$. For every h, let $U_h = U_{1h} + U_{2h}$ where U_{1h} and U_{2h} are the two (non-zero) parts of the sustainable preference U_h that exist according to the representation of sustainable preferences provided in (1) Sect. 2.2 above, established in Chichilnisky (1996a,b). If gains from trade $\mathcal{G}(E)$ were not bounded, there would be a sequence of feasible, individually rational allocations of increasing utility $\{g^j\} = \{g_1^j, \ldots, g_H^j\}_{j=1,2\ldots}$ satisfying (i) $\forall j, \sum_{h=1}^{H} g_h^j = 0$, (ii) $\forall h, j U_h(g_h^{j+1}) > U_h(g_h^j)$ and (iii) for some $k, \lim_{j\to\infty}(U_k(g_k^j)) = \infty$, which implies that $\lim_{j\to\infty} \parallel g_k^j \parallel_\infty = \infty$. Define the set of traders K by $k \in K \iff \lim_{j\to\infty} U_k(g_k^j) = \infty$ so that in particular $\lim_j \parallel g_k^j \parallel = \infty$; then by assumption $K \neq \emptyset$. We show that limited arbitrage contradicts (i),(ii) and (iii) so that gains from trade $\mathcal{G}(E)$ cannot be unbounded with limited arbitrage. By definition of limited arbitrage (2) for $j > j_0$, there exists a p and a j_0 such that $\sum_{h \in K} \langle p, g_h^j \rangle > 0$ for $j > j_0$, because (ii) (iii) imply that $\forall h$, $\{g_h^j\}$ is in $G_h(0)$. However by (i) $\forall j$, $\sum_{h=1}^{H} g_h^j = 0$ so that $\forall p > 0$, $\sum_{h \in J} \langle p, g_h^j \rangle = 0$, a contradiction. The contraction arises from assuming that $\mathcal{G}(E)$ is not bounded. Therefore limited arbitrage implies bounded gains from trade, as we wanted to show.

Next, we derive properties of general sustainable preferences, as stated above. Observe that, under limited arbitrage, when the sequence of purely finitely additive utilities $\{U_{2h}(g_h^j)\}$ in (2) grows without bound as $j \to \infty$, so does the countably additive sequence $\{U_{1h}(g_h^j)\}_{j \to \infty}$ in (2). Assume, to the contrary, that $\{U_{2h}(g_h^j)\}$ grows without bound but $\{U_{1h}(g_h^j)\}$ is bounded. Since as we saw above gains from trade $\sum_h U_h(g_h^j)$ are bounded under limited arbitrage, for each h, $\{U_h(g_h^j)\}_j$ is bounded. However for each j, $U_{2h}^j = U_h^j - U_{1h}^j$ and the right hand side is bounded by assumption, because U_h^j is bounded and we just assumed U_{1h}^j to be bounded as well. Therefore, the sequence $\{U_{2h}(g_h^j)\}_j$ must be bounded, which is a contradiction. Therefore, under the conditions, when the sequence of purely finitely additive utilities $\{U_{2h}(g_h^j)\}_{j \to \infty}$ grows without bound so does the countably additive sequence of utilities $\{U_{1h}(g_h^j)\}$. For each h, consider the sequence of normalized vectors $(\frac{g_h^j}{\|g_h^j\|})$, denoted also $\{g_h^j\}$. We now show that the sequence of normalized vectors $\{g_h^j\}$ has a weak * limit, and that its weak* limit is not zero. First, observe that the normalized sequence $\{g_h^j\}$ is contained in the unit sphere of L_∞, which is weak* compact by Alaoglu's theorem. Consider a subsequence with a weak* limit; we show that this weak* limit is not zero. Since $X = L_\infty$ and utilities are continuous and sustainable, the preferred sets have non-empty interiors and by the properties of sustainable preferences presented in Sect. 2.2, there exist two non-zero prices $p_1 \in L_1$ and $p_2 \in L_\infty^* - L_1$,[6] such that p_1 supports the preferred set of U_{1h} at $\{0\}$, denoted U_{1h}^0, and p_2 supports the preferred set of U_{2h} at $\{0\}$ denoted U_{2h}^0, so that $p = p_1 + p_2$ supports the prefered set of U_h at $\{0\}$, U_h^0. We saw that, under limited arbitrage and with sustainable preferences, for any h, $\lim_j U_{2h}(g_h^j) = \infty$ implies that $\lim_{j \to \infty} U_{1h}(g_h^j) = \infty$, namely when purely finitely additive utility values grow without bound the corresponding countably additive parts do too. This implies in turn that $\forall h$, when the limiting utility values $\lim_j U_h(g_h^j) = \lim_j (U_{1h}(g_h^j) + U_{2h}(g_h^j)) = \infty$, then $\lim_{j \to \infty} U_{1h}(g_h^j) = \infty$, since $\lim_j U_h(g_h^j) = \infty$ and $U_h^j(g_h^j) = U_{1h}(g_h^j) + U_{2h}(g_h^j)$ implies that either $\lim_{j \to \infty} U_{1h}(g_h^j) = \infty$ as we wish to prove, or else $\lim_{j \to \infty} U_{2h}(g_h^j) = \infty$ which in turn implies that $\lim_{j \to \infty} U_{1h}(g_h^j) = \infty$ as seen above. Thus in all cases $\lim_j U_h(g_h^j) \to \infty$ implies $\lim_{j \to \infty} U_{1h}(g_h^j) = \infty$ as we wished to prove. Next, observe that for all h, there exist a subsequence denoted also $\{g_h^j\}$, j_0 and $r > 0$, such that $\langle p_1, g_h^j \rangle \geq r$ when $j > j_0$. Otherwise, $\lim_j \langle p_1, g_h^j \rangle = 0$ and in particular for any $t > 0$, and $\varepsilon > 0$, $\exists j : U_{1h}(g_h^j) > t$ and $(p_1, g_h^j) < \varepsilon$. But $\forall y$

[6] This follows from the initial results of Chichilnisky (1996a,b). $L_\infty^* - L_1$ denotes the complement of L_1 in L_∞^*, namely the space of purely finitely additive measures, see also the Appendix.

satisfying $\langle p_1, y\rangle < 0$, $U_{1h}(y) < U^0_{1h}$ because p_1 supports U^0_{1h}. Therefore, by continuity $\lim_j U_{1h}(g^j_h) \leq 0$, a contradiction since we showed that $\lim_{j\to\infty} U_{1h}(g^j_h) = \infty$. Therefore

$$\forall j > j_0, \quad \left\langle p_1, g^j_h \right\rangle \geq r > 0, \tag{3}$$

which implies that the sequence $\{g^j_h\}$ is weak* bounded away from zero, by definition, since $p_1 \in L_1$. Therefore, we have shown that the weak* compact sequence $\{g^j_h\}$ contains a subsequence, denoted also $\{g^j_h\}$, with a weak* limit denoted g_h, that is not zero because of (3). Consider now the cone C defined by all strictly positive convex combinations of the vectors g_h for all h. Either C is strictly contained in a half-space, or it defines a subspace of X. Since by construction $\sum_{h=1}^H g_h = 0$, C cannot be strictly contained in a half-space. Therefore C defines a subspace. In particular, there is $H' \subset H$, k, and $\forall h \in H'$, $\lambda_h > 0$, such that $(*) - g_k = \sum_{h=1}^{H'} \lambda_h g_h$. □

Corollary 1 *Limited arbitrage is necessary and sufficient for bounded gains from trade in case* (ii).

Proof Consider preferences in *case* (ii), which include linear preferences. Since $G_h = A_h$ in this case as shown in Proposition 1, the set of traders K defined by $k \in K \iff \lim_{j\to\infty} U_k(g^j_k) = \infty$ equals H. In this case bounded gains from trade imply, there can be no sequence $\{g^j_h\}$ satisfying (i), (ii) and (iii) in Proposition 2, so the reciprocal of the statement of this proposition is immediate. Thus limited arbitrage is necessary and sufficient for bounded gains from trade when preferences are in *case* (ii). □

3 Existence of sustainable market equilibrium with short sales

As already mentioned, the markets considered in this article allow unbounded short sales, since the trading domain X is the entire space. Yet, this Section shows that under *limited arbitrage* traders only wish to engage in bounded trades with each other, and the set of efficient trades is compact:

Theorem 1 *Limited arbitrage is necessary and sufficient for the compactness of a non-empty Pareto Frontier* $P(E)$.

Proof This follows Chichilnisky (1991, 1995); Chichilnisky and Heal (1998) and propositions 1 and 2 above. Assume Limited Arbitrage. Observe that the Pareto Frontier is in euclidean space $P(E) \subset R^H_+$. Proposition 2 showed that with limited arbitrage, $P(E)$ is always bounded. To show compactness, it suffices to show that $P(E)$ is closed under limited arbitrage. Without loss of generality, consider a sequence of allocations $\{g^j_h\}_{j=1,2,\ldots}$ satisfying $\forall j$, $\sum_{h=1}^H g^j_h = 0$, so that $\lim_{j\to\infty} \sum_{h=1}^H g^j_h = 0$, and the corresponding utility values $U_1(g^j_1), \ldots, U_H(g^j_H) \in R^H_+$. Assume that the utility values converge either to ∞ or to a utility allocation $V = (V_1, \ldots, V_H) \in R^H_+$ that is undominated by the utility allocation of any other feasible allocation; V may or not be a utility allocation corresponding to a feasible allocation. When limited arbitrage is satisfied, we show that V is the utility allocation corrsponding to a feasible allocation. It suffices to consider the case where the sequence of feasible utility allocations $\{U_1(g_1^j), \ldots, U_H(g_H^j)\}$, and therefore the corresponding allocations $\{g_h^j\}_{j=1,2,\ldots}, h = 1, \ldots, H$ are unbounded. Observe that, as shown in the proof of Proposition 2, the countably additive parts of the utilities $\{U_{11}(g_1^j), \ldots, U_{1H}(g_H^j)\}_{j\to\infty}$ are also unbounded in this case; the normalized sequence $\{\frac{g_h^j}{\|g_h^j\|}\}_j = 1, 2, \ldots,$ denoted also $\{g_h^j\}_{j=1,2,\ldots}$ is weak* precompact by Alaoglu's theorem and as shown in the proof of Proposition 2 it has a weak* convergent subsequence, denoted also $\{g_h^j\}_{j=1,2,\ldots}$ with a non-zero weak* limit $g_h = \lim_{j\to\infty}\{g_h^j\}_{j=1,2,\ldots}$. If $\forall h, g_h^j \notin G_h$ then eventually the utility values of the traders attain their limit for all h, the utility vector V is achieved by a feasible allocation and the proof is complete. It remains to consider the case when for some trader k, $g_k^j \in G_k$; without loss assume that $\forall h, g_k^j \in G_k$. As in Proposition 2, consider the open convex cone C of strictly positive linear combinations of the (non-zero) vectors $g_h, h = 1, 2, \ldots, H, C = \{w = \sum_h \mu_h g_h$ where $\forall h, \mu_h > 0\}$. Either $(a)C$ is contained strictly in a half-space of X or else $(b)C$ is a subspace of X. By construction $\forall j, \sum_h g_h^j = 0$, which eliminates case (a). Therefore case (b) must hold, in particular, there exists $k, g_k \in K$ and $\forall h, \lambda_h \geq 0$ satisfying

$$-g_k = \sum_{h=1}^{H} \lambda_h g_h$$

However, limited arbitrage implies that $\exists p \in \cap_h D_h$ so that $\forall h, \langle p, g_h\rangle \geq 0$, which contradicts $-g_k = \sum_{h=1}^{H} \lambda_h g_h$. The contradiction arises from assuming that the Pareto frontier is not closed under limited arbitrage, therefore $P(E)$ must be closed. Limited arbitrage implies therefore a closed non-empty Pareto Frontier $P(E) \subset R^H$ which, from Proposition 2, is also bounded and hence compact. This establishes sufficiency. The reciprocal is established as follows. Failure of limited arbitrage means as seen above that for any $(U_1(g_1), \ldots, U(g_H)) \in P(E)$, there exists $(v_1, \ldots, v_H)$ satisfying $\forall h, \sum_{h=1}^{H} v_h = 0$ and $U_h(g_h + v_h) > U_h(g_h)$, a contradiction. Therefore limited arbitrage is necessary for a compact non-empty $P(E)$. □

Corollary 2 *Limited arbitrage implies that the Pareto frontier $P(E)$ is homeomorphic to a simplex.*

This follows from Theorem 1, and the convexity of preferences, Arrow and Hahn (1971), Lemma 3, Chapter 5, p. 81.

Theorem 2 *Consider a sustainable market economy $E = \{X, U_h, \Omega_h, h = 1, \ldots, H\}$ where $H \geq 2$, $X = L_\infty$, and $\forall h$, trader h has a sustainable preference U_h. Then the economy E has a sustainable market equilibrium if and only if it satisfies limited arbitrage, and the sustainable equilibrium is Pareto efficient.*

Proof Necessity first. Without loss assume that $\forall h, \Omega_h = 0 \in X$. Let p^* be a price equilibrium and let $f^* = (f_1^*, \ldots, f_H^*)$ be the corresponding equilibrium allocation.

If limited arbitrage fails, $\exists h$ and $\{g^j\} \in G_h$ such that $\langle p^*, g^j \rangle \leq 0$ for some $j > j_0$ namely g^j is affordable at prices p^*. Recall that G_h is the same for every allocation by Proposition 1. It follows that $\exists j_0 > 0$ such that for $j > j_0$, $U_h(f_h^* + g^j) > U_h(f_h^*)$ which, together with the affordability of g^j, contradicts the fact that f^* is an equilibrium allocation. Limited arbitrage is therefore necessary for existence of an equilibrium.

For sufficiency, Theorem 1 established that the Pareto frontier is homeomorphic to a simplex when limited arbitrage is satisfied. The standard Negishi fixed point argument on the Pareto frontier $P(E)$ in utility space R^H establishes therefore the existence of a pseudoequilibrium, see Negishi (1960) and Chichilnisky and Heal (1984). To complete the proof, observe that $\forall h = 1, 2, \ldots, H$ there exists always an allocation in X of strictly lower value than the pseudo equilibrium f_h^* at the price p^*. Therefore by Lemma 3, Chapter 4, p. 81 of Arrow and Hahn (1971) the quasi-equilibrium $\langle p^*, g^* \rangle$ is also a competitive equilibrium, completing the proof of existence. Pareto efficiency follows from the fact that the equilibrium is in the Pareto frontier $P(E)$. □

4 Market value and the axiom of choice

The existence of a sustainable equilibrium ensures the logic consistency of the concept of sustainable markets introduced here. The main condition required for existence is limited arbitrage, a condition that has been used to prove existence in the literature (Chichilnisky 1991, 1995, 1994a,b, 1996c,d, 1998; Chichilnisky and Heal 1998), applied in this case to markets where the traders have sustainable preferences. The notion of an equilibrium price in a sustainable market requires however further discussion. An equilibrium price is defined here—as is usual—namely a continuous linear function on commodities or trades, and this price establishes the economic value of commodities at a market equilibrium. The space of prices is here L_∞^*, the dual of the space of commodities L_∞ that has been characterized (see the Appendix) as consisting of the linear sum of two subspaces, one subspace consisting of prices in L_1 that have a ready interpretation, and the second subspace consisting of finitely additive measures on R that require further explanation. Since preferences are sustainable, a price equilibrium will have the same form as a sustainable preference as characterized in Sect. 2.2 above and in Chichilnisky (1996a,b), namely a convex combination of a path of prices through time that is an element of L_1 and of a purely finitely additive measure, for example, a measure that focuses its weight on unbounded sets in R. In this context, therefore, a market price may assign two types of economic values: (i) an instantaneous value to commodities through time, and in addition (ii) a value to the long-run future. The second term (ii) may seem unusual in standard markets, but it seems entirely appropriate for a sustainable market equilibrium. It modifies the conventional notion of prices just enough to value the long run, as seems required for sustainable market solutions.

The finitely additive part of the price that assigns value to the long-run future establishes a connection between sustainable markets and the Axiom of Choice in the foundation of mathematics, which postulates that there exists a universal and consistent fashion to select an element from every set; see Dunford and Schwartz

(1958), Yosida (1974), Yosida and Hewitt (1952) (20, 21), Chichilnisky and Heal (1997), Kadane and O'Hagan (1995), Purves and Sudderth (1976), Dubins (1975) and Dubins and Savage (1965). It is possible to illustrate—but not in general construct—a purely finitely additive measure on R, or on any finite open interval (a, b) of R, see examples in Chichilnisky (2010a,b). The issue of constructibility is not unique to sustainable markets; it is an issue shared by the proof of the second fundamental theorem of welfare economics, see Debreu (1953), which requires Hahn-Banach Theorem and therefore requires also the Axiom of Choice. Therefore the prices in the second welfare theorem are also not constructible. The proof of existence of such purely finitely additive functions can be achieved in various ways but each requires the Axiom of Choice or a related result. To illustrate the problem, consider the function $\phi(g(t)) = \lim_{t\to\infty} g(t)$ that is defined only on a closed strict subspace L'_∞ of L_∞ consisting of functions that have a limit at infinity. This function is continuous and linear on L'_∞. One can use Hahn-Banach's theorem to extend this function ϕ from the closed subspace $L'_\infty \subset L_\infty$ to the entire space L_∞ preserving the norm. Since the extension is not in L_1, it defines a purely finitely additive measure, as shown in the Appendix. However, in a general form, Hahn-Banach's theorem requires the Axiom of Choice, which has been shown to be independent from the rest of the axioms of Mathematics (Godel 1940). Alternatively, one can extend the notion of a *limit* to encompass all functions in L_∞ including those with no standard limit. This can be achieved by defining convergence along a *free ultrafilter* arising from Stone-Cech compactification of the real line R as in Chichilnisky and Heal (1997). However, the existence of a *free ultrafilter* requires once again the Axiom of Choice (Godel 1940). This illustrates why the actual construction of a *purely finitely additive measure* requires the Axiom of Choice. Since sustainable markets have prices that include purely finitely additive measures, this provides a connection between the Axiom of Choice and sustainable markets. It appears that the consideration of sustainable goals about consumption in the long-run future conjures up the Axiom of Choice that is independent from the rest of mathematics (Godel 1940).

Appendix

Example A preference that is insensitive to the present

Consider $W(f) = \lim\inf_{x\epsilon R}(f(x))$. This utility is insensitive to the present and therefore does not satisfy Axiom 2. In addition, this map is not linear, failing Axiom 1.

The dual space L^*_∞: countably additive and purely finitely additive measures

See Yosida (1974); Yosida and Hewitt (1952); Dunford and Schwartz (1958). A measure η is called *finitely additive* when for any family of pairwise disjoint measurable sets $\{A_i\}_{i=1,\ldots,N}\eta(\bigcup_{i=1}^N A_i) = \sum_{i=1}^N \eta(A_i)$. The measure η is called *countably additive* when for any countable family of pairwise disjoint measurable sets $\{A_i\}_{i=1,\ldots,\infty}\eta(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \eta(A_i)$. The space of continuous linear functions on L_∞ is the 'dual' of L_∞ and is denoted L^*_∞. This space has been characterized, e.g., in Yosida and Hewitt (1952); Yosida (1974). $L^*_\infty = L_1 + (L^1_\infty - L_1)$: it consists both of L_1 functions g that define countably additive measures ν on R by the rule $\nu(A) = \int_A g(x)dx$ where $\int_R \mid g(x) \mid dx < \infty$ so that υ is *absolutely continuous* with respect to the Lebesgue measure, plus measures that are not countably additive, also called purely finitely additive measures, forming a subspace denoted $L^*_\infty - L_1$. While a countable measure can be identified with an L_1 function, namely its so called 'density', purely finitely additive measures cannot be identified by such functions.

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