[b]For historical perspectives regarding option pricing and hedging, *see* **Black, Fischer**; **Merton, Robert C.**; **Arbitrage: Historical Perspectives**; and **Option Pricing Theory: Historical Perspectives**. For a more thorough quantitative treatment, *see* **Risk-neutral Pricing**.
[c]Is this bold distancing from normality of mathematical finance professors, clearly implied from the authors of [2], a decisive step toward illuminating the perception they have of their own personalities? Or is it just a gimmick used to add another humorous ingredient to the joke? The answer is left for the reader to determine.

References

[1] Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities, *The Journal of Political Economy* **81**, 637–654.
[2] Delbaen, F. & Schachermayer, W. (2006). *The Mathematics of Arbitrage*, *Springer Finance*, Springer-Verlag, Berlin.
[3] Merton, R.C. (1973). Theory of rational option pricing, *Bell Journal of Economics and Management Science* **4**, 141–183.

Further Reading

Dalang, R.C., Morton, A. & Willinger, W. (1990). Equivalent martingale measures and no-arbitrage in stochastic securities market models, *Stochastics and Stochastics Reports* **29**, 185–201.
Delbaen, F. (1992). Representing martingale measures when asset prices are continuous and bounded, *Mathematical Finance* **2**, 107–130.
Delbaen, F. & Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing, *Mathematische Annalen* **300**, 463–520.
Delbaen, F. & Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes, *Mathematische Annalen* **312**, 215–250.
Elworthy, K.D., Li, X.-M. & Yor, M. (1999). The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes, *Probability Theory and Related Fields* **115**, 325–355.
Föllmer, H. & Schied, A. (2004). *Stochastic Finance*, *de Gruyter Studies in Mathematics*, extended Edition, Walter de Gruyter & Co., Berlin, Vol. 27.
Hull, J.C. (2008). *Options, Futures, and Other Derivatives*, 7th Edition, Prentice Hall.
Michael Harrison, J. & Kreps, D.M. (1979). Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* **20**, 381–408.
Michael Harrison, J. & Pliska, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Applications* **11**, 215–260.
Shreve, S.E. (2004). *Stochastic Calculus for Finance. I: The Binomial Asset Pricing Model*, *Springer Finance*, Springer-Verlag, New York.

Related Articles

Black, Fischer; **Equivalent Martingale Measures**; **Fundamental Theorem of Asset Pricing**; **Free Lunch**; **Good-deal Bounds**; **Merton, Robert C.**; **Ross, Stephen**; **Risk-neutral Pricing**.

CONSTANTINOS KARDARAS

ARCH *see* GARCH Models

ARMA *see* Autoregressive Moving Average (ARMA) Processes

Arrow, Kenneth

Most financial decisions are made under conditions of uncertainty. Yet a formal analysis of markets under uncertainty emerged only recently, in the 1950s. The matter is complex as it involves explaining how individuals make decisions when facing uncertain situations, the behavior of market instruments such as insurance, securities, and their prices, the welfare properties of the distribution of goods and services under uncertainty, and how risks are shared among the traders. It is not even obvious how to formulate market clearing under conditions of uncertainty. A popular view in the middle of the last century was that markets would only clear on the average and asymptotically in large economies.[a] This approach was a reflection of how insurance markets work, and followed a notion of actuarially fair trading.

A different formulation was proposed in the early 1950s by Arrow and Debreu [10, 12, 30]. They introduced an economic theory of markets in which the treatment of uncertainty follows basic principles of physics. The contribution of Arrow and Debreu is as fundamental as it is surprising. For Arrow and Debreu, markets under uncertainty are formally identical to markets without uncertainty. In their approach, uncertainty all but disappears.[b]

It may seem curious to explain trade with uncertainty as though uncertainty did not matter. The disappearing act of the issue at stake is an unusual way to think about financial risk, and how we trade when facing such risks. But the insight is valuable. Arrow and Debreu produced a rigorous, consistent, general theory of markets under uncertainty that inherits the most important properties of markets without uncertainty. In doing so, they forced us to clarify what is intrinsically different about uncertainty.

This article summarizes the theory of markets under uncertainty that Arrow and Debreu created, including critical issues that arise from it, and also its legacy. It focuses on the way Arrow introduced securities: how he defined them and the limits of his theory. It mentions the theory of insurance that Arrow pioneered together with Malinvaud and others [6], as well as the theory of risk bearing that Arrow developed on the basis of expected utility [7], following the axioms of Von Neumann and Morgenstern [41], Hernstein and Milnor [33], De Groot [31], and Villegas [40]. The legacy of Arrow's work is very extensive and some of it surprising. This article describes his legacy along three lines: (i) individual and idiosyncratic risks, (ii) rare risks and catastrophic events, and (iii) endogenous uncertainty.

Biographical Background

Kenneth Joseph Arrow is American economist and joint winner of the Nobel Memorial Prize in Economics with John Hicks in 1972. Arrow taught at Stanford University and Harvard University. He is one of the founders of modern (post World War II) economic theory, and one of the most important economists of the twentieth century. For a full biographical note, the reader is referred to [18]. Born in 1921 in New York City to Harry and Lilian Arrow, Kenneth was raised in the city. He graduated from Townsend Harris High School and earned a bachelor's degree from the City College of New York studying under Alfred Tarski. After graduating in 1940, he went to Columbia University and after a hiatus caused by World War II, when he served with the Weather Division of the Army Air Forces, he returned to Columbia University to study under the great statistician Harold Hotelling at Columbia University. He received a master's degree in 1941 studying under A. Wald, who was the supervisor of his master's thesis on stochastic processes. From 1946 to 1949 he spent his time partly as a graduate student at Columbia and partly as a research associate at the Cowles Commission for Research in Economics at the University of Chicago; it was in in Chicago that he met his wife Selma Schweitzer. During that time, he also held the position of Assistant Professor of Economics at the University of Chicago. Initially interested in following a career as an actuary, in 1951 he earned his doctorate in economics from Columbia University working under the supervision of Harold Hotelling and Albert Hart. His published work on risk started in 1951 [3]. In developing his own approach to risk, Arrow grapples with the ideas of Shackle [39], Knight [35], and Keynes [34] among others, seeking and not always finding a rigorous mathematical foundation. His best-known works on financial markets date back to 1953 [3]. These works provide a solid foundation based on the role of securities in the allocation of risks [4, 5, 7, 9, 10]. His approach can be described as a state contingent security approach to the allocations of risks in an economy, and is largely an extension of the same approach he followed in his work on general equilibrium theory with Gerard Debreu, for which he was awarded the Nobel Prize in 1972 [8]. Nevertheless, his work connects also with social issues of risk allocation and with the French literature of the time, especially [1, 2].

Markets under Uncertainty

The Arrow–Debreu theory conceptualizes uncertainty with a number of possible states of the world $s = 1, 2, \ldots$ that may occur. Commodities can be in one of several states, and are traded separately in each of the states of nature. In this theory, one does not trade a good, but a "contingent good", namely, a good in each state of the world: apples when it rains and apples when it shines [10, 12, 30]. This way the theory of markets with N goods and S states of nature is formally identical to the theory of markets without uncertainty but with $N \times S$ commodities. Traders trade "state contingent commodities". This simple formulation allows one to apply the results of the theory of markets without uncertainty, to markets with uncertainty. One recovers most of the important results such as (i) the existence of a market equilibrium and (ii) the "invisible hand theorem" that establishes that market solutions are always Pareto efficient. The approach is elegant, simple, and general.

Along with its elegance and simplicity, the formulation of this theory can be unexpectedly demanding. It requires that we all agree on all the possible states of the world that describe "collective uncertainty", and that we trade accordingly. This turns out to be more demanding than it seems: for example, one may need to have a separate market for apples when it rains than when it does not, and separate market prices for each case. The assumption requires $N \times S$ markets to guarantee market efficiency, a requirement that in some cases militates against the applicability of the theory. In a later article, Arrow simplified the demands of the theory and reduced the number of markets needed for efficiency by defining "securities", which are different payments of money exchanged among the traders in different states of nature [4, 5]. This new approach no longer requires trading "contingent" commodities but rather trading a combination of commodities and securities. Arrow proves that by trading commodities and securities, one can achieve the same results as trading state contingent commodities [4, 5]. Rather than needing $N \times S$ markets, one needs a fewer number of markets, namely, N markets for commodities and $S - 1$ markets for securities. This approach was a great improvement and led to the study of securities in a rigorous and productive manner, an area in which his work has left a large legacy. The mathematical requirement to reach Pareto efficiency was simplified gradually to require that the securities traded should provide for each trader a set of choices with the same dimension as the original state contingent commodity approach. When this condition is not satisfied, the markets are called "incomplete". This led to a large literature on incomplete markets, for example, [26, 32], in which Pareto efficiency is not assured, and government intervention may be required, an area that exceeds the scope of this article.

Individual Risk and Insurance

The Arrow–Debreu theory is not equally well suited for all types of risks. In some cases, it could require an unrealistically large number of markets to reach efficient allocations. A clear example of this phenomenon arises for those risks that pertain to one individual at a time, called *individual risks*, which are not readily interpreted as states of the world on which we all agree and are willing to trade. Individuals' accidents, illnesses, deaths, and defaults, are frequent and important risks that fall under this category. Arrow [6] and Malinvaud [37] showed how individual uncertainty can be reformulated or reinterpreted as collective uncertainty. Malinvaud formalized the creation of states of collective risks from individual risks, by lists that describe all individuals in the economy, each in one state of individual risk. The theory of markets can be reinterpreted accordingly [14, 37, 38], yet remains somewhat awkward. The process of trading under individual risk using the Arrow–Debreu theory requires an unrealistically large number of markets. For example with N individuals, each in one of two individual states G (good) and B (bad), the number of (collective) states that are required to apply the Arrow–Debreu theory is $S = 2^N$. The number of markets required is as above, either $S \times N$ or $N + S - 1$. But with $N = 300$ million people, as in the US economy, applying the Arrow–Debreu approach would require $N \times S = N \times 2^{300 \text{ million}}$ markets to achieve Pareto efficiency, more markets than the total amount of particles in the known universe [25]. For this reason, individual uncertainty is best treated with another formulation of uncertainty involving individual states of uncertainty and insurance rather than securities, in which market clearing is defined on the average and may never actually occur. In this new approach, instead of requiring $N + S - 1$ markets, one requires only N commodity markets and, with two states of individual risk, just one security: an insurance contract suffices to obtain asymptotic efficiency [37, 38]. This is a satisfactory theory of individual risk and insurance, but it leads only to asymptotic market clearing and Pareto efficiency. More recently, the theory was improved and it was shown that one can obtain exact market-clearing solutions and Pareto-efficient allocations based on N commodity markets with the introduction of a limited number of financial instruments called *mutual insurance* [14]. It is shown in [14] that if there are N households (consisting of H types), each facing the possibility of being in S individual states together with T collective states, then ensuring Pareto optimality requires only $H(S-1)T$ independent mutual insurance policies plus T pure Arrow securities.

Choice and Risk Bearing

Choice under uncertainty explains how individuals rank risky outcomes. In describing how we rank choices under uncertainty, one follows principles that were established to describe the way nature ranks what is most likely to occur, a topic that was widely explored and is at the foundation of statistics [31, 40]. To explain how individuals choose under conditions of uncertainty, Arrow used behavioral axioms that were introduced by Von Neumann and Morgenstern [41] for the theory of games[c] and axioms defined by De Groot [31] and Villegas [40] for the foundation of statistics. The main result obtained in the middle of the twentieth century was that under rather simple behavioral assumptions, individuals behave as though they were optimizing an "expected utility function". This means that they behave as though they have (i) a utility u for commodities, which is independent of the state of nature, and (ii) subjective probabilities about how likely are the various states of nature. Using the classic axioms one constructs a ranking of choice under uncertainty obtaining a well-known expected utility approach. Specifically, traders choose over "lotteries" that achieve different outcomes in different states of nature. When states of nature and outcomes are represented by real numbers in R, a lottery is a function $f : R \to R^N$, a utility is a function $u : R^N \to R$, and a subjective probability is $p : R \to [0, 1]$ with $\int_R p(s) = 1$. Von Neumann, Arrow, and Hernstein and Milnor, all obtained the same classic "representation theorem" that identifies choice under uncertainty by the ranking of lotteries according to a real-valued function W, where W has the now familiar "expected utility" form:

$$W(f) = \int_{s \in R} p(s).u(f(s))\,\mathrm{d}s \tag{1}$$

The utility function u is typically bounded to avoid paradoxical behavior. The expected utility approach just described has been generally used since the mid-twentieth century. Despite its elegance and appeal, from the very beginning, expected utility has been unable to explain a host of experimental evidence that was reported in the work of Allais [2] and others. There has been a persistent conflict between theory and observed behavior, but no axiomatic foundation to replace Von Neumann's foundational approach. The reason for this discrepancy has been identified more recently, and it is attributed to the fact that expected utility is dominated by frequent events and neglects rare events—even those that are potentially catastrophic, such as widespread default in today's economies. That expected utility neglects rare events was shown in [17, 19, 23]. In [23], the problem was traced back to Arrow's axiom of monotone continuity [7], which Arrow attributed to Villegas [40], and to the corresponding continuity axioms of Hernstein and Milnor, and De Groot [31], who defined a related continuity condition denoted "SP_4". Because of this property, on which Arrow's work is based, the expected utility approach has been characterized as the "dictatorship" of frequent events, since it is dominated by the consideration of "normal" and frequent events [19]. To correct this bias, and to represent more accurately how we choose under uncertainty, and to arrive at a more realistic meaning of rationality, a new axiom was added in [17, 19, 21], requiring equal treatment for frequent and for rare events. The new axiom was subsequently proven to be the logic negation of Arrow's monotone continuity that was shown to neglect small probability events [23].

The new axioms led to a "representation theorem" according to which the ranking of lotteries is a modified expected utility formula

$$W(f) = \int_{s \in R} p(s).u(f(s))\,\mathrm{d}s + \phi(f) \qquad (2)$$

where ϕ is a continuous linear function on lotteries defined by a finite additive measure, rather than a countably additive measure [17, 19]. This measure assigns most weight to rare events. The new formulation has both types of measures, so the new characterization of choice under uncertainty incorporates both (i) frequent and (ii) rare events in a balanced manner, conforming more closely to the experimental evidence on how humans choose under uncertainty [15]. The new specification gives well-deserved importance to catastrophic risks, and a special role to fear in decision making [23], leading to a more realistic theory of choice under uncertainty and foundations of statistics, [15, 23, 24]. The legacy of Kenneth Arrow's work is surprising but strong: the new theory of choice under uncertainty coincides with the old when there are no catastrophic risks so that, in reality, the latter is an extension of the former to incorporate rare events. Some of the most interesting applications are to environmental risks such as global warming [25]. Here Kenneth Arrow's work was prescient: Arrow was a contributor to the early literature on environmental risks and irreversibilities [11], along with option values.

Endogenous Uncertainty and Widespread Default

Some of the risks we face are not created by nature. They are our own creation, such as global warming or the financial crisis of 2008 and 2009 anticipated in [27]. In physics, the realization that the observer matters, that the observer is a participant and creates uncertainty, is called *Heisenberger's uncertainty principle*. The equivalent in economics is an uncertainty principle that describes how we create risks through our economic behavior. This realization led to the new concept of "markets with endogenous uncertainty", created in 1991, and embodied in early articles [16, 27, 28] that established some of the basic principles and welfare theorems in markets with endogenous uncertainty. This, and other later articles ([20, 25, 27, 36]), established basic principles of existence and the properties of the general equilibrium of markets with endogenous uncertainty. It is possible to extend the Arrow–Debreu theory of markets to encompass markets with endogenous uncertainty and also to prove the existence of market equilibrium under these conditions [20]. But in the new formulation, Heisenberg's uncertainty principle rears its quizzical face. It is shown that it is no longer possible to fully hedge the risks that we create ourselves [16], no matter how many financial instruments we create. The equivalent of Russel's paradox in mathematical logic appears also in this context due to the self-referential aspects of endogenous uncertainty [16, 20]. Pareto efficiency of equilibrium can no longer be ensured. Some of the worst economic risks we face are endogenously determined—for example, those that led to the 2008–2009 global financial crisis [27]. In [27] it was shown that the creation of financial instruments to hedge individual risks—such as credit default insurance that is often a subject of discussion in today's financial turmoil—by themselves induce collective risks of widespread default. The widespread default that we experience today was anticipated in [27], in 1991, and in 2006, when it was attributed to endogenous uncertainty created by financial innovation as well as to our choices of regulation or deregulation of financial instruments. Examples are the extent of reserves that are required for investment banking operations, and the creation of mortgage-backed securities that are behind many of the default risks faced today [29]. Financial innovation of this nature, and the attendant regulation of new financial instruments, causes welfare gains for individuals—but at the same time creates new risks for society that bears the collective risks that ensue, as observed in 2008 and 2009. In this context, an extension of the Arrow–Debreu theory of markets can no longer treat markets with endogenous uncertainty as equivalent to markets with standard commodities. The symmetry of markets with and without uncertainty is now broken. We face a brave new world of financial innovation and the endogenous uncertainty that we create ourselves. Creation and hedging of risks are closely linked, and endogenous uncertainty has acquired a critical role in market performance and economic welfare, an issue that Kenneth Arrow has more recently tackled himself through joint work with Frank Hahn [13].

Acknowledgments

Many thanks are due to Professors Rama Cont and Perry Mehrling of Columbia University and Barnard College, respectively, for their comments and excellent suggestions.

End Notes

[a] See [37, 38]; later on Werner Hildenbrand followed this approach.
[b] They achieved the same for their treatment of economic dynamics. Trading over time and under conditions of uncertainty characterizes financial markets.
[c] And similar axioms used by Hernstein and Milnor [33].
[d] Specifically to avoid the so-called St. Petersburg paradox, see [7].

References

[1] Allais, M. (ed) (1953). *Fondements el Applications de la Theorie du Risque en Econometrie*, CNRS, Paris.

[2] Allais, M. (1987). The general theory of random choices in relation to the invariant cardinality and the specific probability function, in *Risk Decision and Rationality*, B.R. Munier, ed., Reidel, Dordrech The Netherlands, pp. 233–289.

[3] Arrow, K. (1951). Alternative approaches to the theory of choice in risk – taking situations, *Econometrica* **19**(4), 404–438.

[4] Arrow, K. (1953). Le Role des Valeurs Boursiers pour la Repartition la Meilleure des Risques, *Econometrie* **11**, 41–47. Paris CNRS, translated in English in *RES* 1964 (below).

[5] Arrow, K. (1953). The role of securities in the optimal allocation of risk bearing, *Proceedings of the Colloque sur les Fondaments et Applications de la Theorie du Risque en Econometrie*. CNRS, Paris. English Transation published in *The Review of Economic Studies* Vol. 31, No. 2, April 1964, p. 91–96.

[6] Arrow, K. (1953). Uncertainty and the welfare economics of medical care, *American Economic Review* **53**, 941–973.

[7] Arrow, K. (1970). *Essays on the Theory of Risk Bearing*, North Holland, Amsterdam.

[8] Arrow, K. (1972). *General Economic Equilibrium: Purpose Analytical Techniques Collective Choice*, Les Prix Nobel en 1972, Stockholm Nobel Foundation pp. 253–272.

[9] Arrow, K. (1983). *Collected Papers of Kenneth Arrow*, Belknap Press of Harvard University Press.

[10] Arrow, K.J. & Debreu, G. (1954). Existence of an equilibrium for a competitive economy, *Econometrica* **22**, 265–290.

[11] Arrow, K.J. & Fischer, A. (1974). Environmental preservation, uncertainty and irreversibilities, *Quarterly Journal of Economics* **88**(2), 312–319.

[12] Arrow, K. & Hahn, F. (1971). *General Competitive Analysis*, Holden Day, San Francisco.

[13] Arrow, K. & Hahn, F. (1999). Notes on sequence economies, transaction costs and uncertainty, *Journal of Economic Theory* **86**, 203–218.

[14] Cass, D., Chichilnisky, G. & Wu, H.M. (1996). Individual risk and mutual insurance, *Econometrica* **64**, 333–341.

[15] Chanel, O. & Chichilnisky, G. (2009). The influence of fear in decisions: experimental evidence, *Journal of Risk and Uncertainty* **39**(3).

[16] Chichilnisky, G. (1991, 1996). Markets with endogenous uncertainty: theory and policy, *Columbia University Working paper 1991 and Theory and Decision* **41**(2), 99–131.

[17] Chichilnisky, G. (1996). Updating Von Neumann Morgernstern axioms for choice under uncertainty with catastrophic risks. *Proceedings of Conference on Catastrophic Risks, Fields Institute for Mathematical Sciences*, Toronto, Canada.

[18] Chichilnisky, G. (ed) (1999). *Markets Information and Uncertainty: Essays in Honor of Kenneth Arrow*, Cambridge University Press.

[19] Chichilnisky, G. (2000). An axiomatic treatment of choice under uncertainty with catastrophic risks, *Resource and Energy Economics* **22**, 221–231.

[20] Chichilnisky, G. (1999/2008). Existence and optimality of general equilibrium with endogenous uncertainty, in *Markets Information and Uncertainty: Essays in Honor of Kenneth Arrow*, 2nd Edition, G. Chichilnisky, ed., Cambridge University Press, Chapter 5.

[21] Chichilnisky, G. (2009). The foundations of statistics with Black Swans, *Mathematical Social Sciences*, DOI:10.1016/j.mathsocsci.2009.09.007.

[22] Chichilnisky, G. (2009). The limits of econometrics: non parametric estimation in Hilbert spaces, *Econometric Theory* **25**, 1–17.

[23] Chichilnisky, G. (2009). "The Topology of Fear" invited presentation at NBER conference in honor of Gerard Debreu, UC Berkeley, December 2006, *Journal of Mathematical Economics* **45**(11–12), December 2009. Available online 30 June 2009, ISSN 0304–4068, DOI: 10.1016/j.jmateco.2009.06.006.

[24] Chichilnisky, G. (2009a). Subjective Probability with Black Swans, *Journal of Probability and Statistics* (in press, 2010).

[25] Chichilnisky, G. & Heal, G. (1993). Global environmental risks, *Journal of Economic Perspectives, Special Issue on the Environment* Fall, 65–86.

[26] Chichilnisky, G. & Heal, G. (1996). On the existence and the structure pseudo-equilibrium manifold, *Journal of Mathematical Economics* **26**, 171–186.
[27] Chichilnisky, G. & Wu, H.M. (1991, 2006). General equilibrium with endogenous uncertainty and default, Working Paper Stanford University, 1991, *Journal of Mathematical Economics* **42**, 499–524.
[28] Chichilnisky, G., Heal, G. & Dutta, J. (1991). *Endogenous Uncertainty and Derivative Securities in a General Equilibrium Model*, Working Paper Columbia University.
[29] Chichilnisky, G., Heal, G. & Tsomocos, D. (1995). Option values and endogenous uncertainty with asset backed securities, *Economic Letters* **48**(3–4), 379–388.
[30] Debreu, G. (1959). *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, John Wiley & Sons, New York.
[31] De Groot, M.H. (1970, 2004). *Optimal Statistical Decisions*, John Wiley & Sons, Hoboken New Jersey.
[32] Geanakopolos, J. (1990). An introduction to general equilibrium with incomplete asset markets, *Journal of Mathematical Economics* **19**, 1–38.
[33] Hernstein, N. & Milnor, J. (1953). An axiomatic approach to measurable utility, *Econometrica* **21**, 219–297.
[34] Keynes, J.M. (1921). *A Treatise in Probability*, MacMillan and Co., London.
[35] Knight, F. (1921). *Risk Uncertainty and Profit*, Houghton Miffin and Co., New York.
[36] Kurz, M. & Wu, H.M. (1996). Endogenous uncertainty in a general equilibrium model with price - contingent contracts, *Economic Theory* **6**, 461–488.
[37] Malinvaud, E. (1972). The allocation of individual risks in large markets, *Journal of Economic Theory* **4**, 312–328.
[38] Malinvaud, E. (1973). Markets for an exchange economy with individual; Risks, *Econometrica* **41**, 383–410.
[39] Shackle, G.L. (1949). *Expectations in Economics*, Cambridge University Press, Cambridge, UK.
[40] Villegas, C. (1964). On quantitiative probability $\sigma-$ algebras, *Annals of Mathematical Statistics* **35**, 1789–1800.
[41] Von Neumann, J. & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

Related Articles

Arrow–Debreu Prices; **Risk Aversion**; **Risk Premia**; **Utility Theory: Historical Perspectives**.

Graciela Chichilnisky

Arrow–Debreu Prices

Arrow–Debreu prices are the prices of "atomic" time and state-contingent claims, which deliver one unit of a specific consumption good if a specific uncertain state realizes at a specific future date. For instance, claims on the good "ice cream tomorrow" are split into different commodities depending on whether the weather will be good or bad, so that good-weather and bad-weather ice cream tomorrow can be traded separately. Such claims were introduced by Arrow and Debreu in their work on general equilibrium theory under uncertainty, to allow agents to exchange state and time contingent claims on goods. Thereby the general equilibrium problem with uncertainty can be reduced to a conventional one without uncertainty. In finite-state financial models, Arrow–Debreu securities delivering one unit of the numeraire good can be viewed as natural atomic building blocks for all other state–time contingent financial claims; their prices determine a unique arbitrage-free price system.

Arrow–Debreu Equilibrium Prices

This section explains Arrow–Debreu prices in an equilibrium context, where they originated, see [1, 3]. We first consider a single-period model with uncertain states that will be extended to multiple periods later. For this exposition, we restrict ourselves to a single consumption good only, and consider a pure exchange economy without production.

Let $(\Omega, \mathcal{F})$ be a measurable space of finitely many outcomes $\omega \in \Omega = \{1, 2, \ldots, m\}$, where the σ-field $\mathcal{F} = 2^{\Omega}$ is the power set of all events $A \subset \Omega$. There is a finite set of agents, each seeking to maximize the utility $u^a(c^a)$ from his or her consumption $c^a = (c_0^a, c_1^a(\omega)_{\omega\in\Omega})$ at present and future dates 0 and 1, given some endowment that is denoted by a vector $(e_0^a, e_1^a(\omega)) \in \mathbb{R}_{++}^{1+m}$. For simplicity, let consumption preferences of agent a be of the expected utility form

$$u^a(c^a) = U_0^a(c_0) + \sum_{\omega=1}^{m} P^a(\omega) U_\omega^a(c_1(\omega)) \qquad (1)$$

where $P^a(\omega) > 0$ are subjective probability weights, and the direct utility functions U_ω^a and U_0^a are, for present purposes, taken to be of the form $U_i^a(c) =$