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# GENERAL EQUILIBRIUM WITH UNCERTAINTY THE WORK OF KENNETH ARROW

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ABSTRACT. This article summarizes the theory of markets under uncertainty that Arrow and Debreu created, its achievements and the critical issues that it raises. It focuses on the way Arrow introduced securities: how he defined them and what were the most useful developments of this theory. It mentions the theory of insurance that Arrow pioneered together with Malinvaud and others, and Arrow's theory of risk bearing that follows the axiomatic treatement of Von Neumann and Morgenstern, Milnor, Hernstein, De Groot, and Villegas. The classic expected utility theory embraces normal or frequent risks at the expense of neglecting rare events, even those with important consequences such as catastrophes, causing paradoxical experimental behavior as shown by the author. Kenneth Arrow encouraged the creation of an extension of classic axiomatic theory, and in the early 1990s the author proposed axioms that treat symmetrically frequent and rare events, leading to a new definition of rationality and a new form of choice under uncertainty that conforms better to the experimental evidence. Additionally, the introduction of markets with 'endogenous uncertainty' in the early 1990's challenged the foundation of Arrow Debreu's theory of markets. Endogenous uncertainty is generated by the workings of the economy, and breaks down the equivalence between markets with and without uncertainty that prevails in Arrow Debreu's theory. This leads to a new world of uncertainty creation through financial innovation and regulation that is at the heart of the financial markets turmoil of 2008. In 1999 Arrow contributed to the theory of endogenous uncertainty in his work with Frank Hahn.

## 1. Introduction

Most financial decisions are made under conditions of uncertainty. Yet a formal analysis of markets under uncertainty was achieved relatively recently, in the 1950's. The matter is complex as it involves explaining how individuals make decisions under uncertainty, market instruments such as securities and their prices, welfare properties of the distribution of goods and services, and how risks are shared among the traders. It is not even obvious how to formulate market clearing under conditions of uncertainty. A popular approach in the middle of last century was that markets would only clear on the average and asymptotically in large economies,

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Malinvaud.[1] This approach was a reflection of how insurance markets work, and followed a notion of actuarially fair trading.

A different formulation was proposed in the early 1950's by Kenneth Arrow and Gerard Debreu [5], [27] [9]. They introduced an economic theory of markets where the treatment of uncertainty follows basic principles of physics.isCommodities are in one of several states of nature, as particles are in quantum mechanics. The contribution of Arrow and Debreu is as fundamental as it is surprising. For Arrow and Debreu, markets under uncertainty are formally identical to markets without uncertainty. In their approach, uncertainty all but disappears.[2]

It may seem curious to explain trade with uncertainty as if uncertainty did not matter. The disappearing act of the issue at stake is an unusual way to think about financial risk, and how we trade when facing such risks. But the insight is valuable. Arrow and Debreu produced a rigorous, consistent, general theory of markets under uncertainty that inherits the most important properties of markets without uncertainty. In doing so, they forced us to clarify what is intrinsically different about uncertainty.

This article summarizes the theory of markets under uncertainty that Arrow and Debreu created, including critical issues that arise from it and its legacy. It focuses on the way Arrow introduced securities: how he defined them and the limits of his theory. It mentions the theory of insurance that Arrow pioneered together with Malinvaud and others [11], as well as the theory of risk bearing that Arrow developed based on expected utility [6], following the axioms of Von Neumann and Morgenstern [38], Hernstein and Milnor [32], De Groot [29], and Villegas [37]. Yet expected utility theory was troubled by experimental paradoxes since its very beginnings [1]. More recently it was discovered that expected utility focuses excessively on 'normal' or frequent risks at the expense of neglecting rare events, even those having important consequences such as catastrophes [20] [21]. This conflicts with the way rational individuals behave when confronted with risks that inspire fear. At the invitation of Kenneth Arrow an extension of the notion of rationality was created by introducing a new axiom: equal treatment for frequent and for rare events and the new axiom, which requires sensitivity to rare events, was curiously found to be the logical negation of Arrow's Monotone Continuity Axiom in [6], cf. Chichilnisky [20] [21] [25] . The new theory of individual choice under uncertainty is an extension of classic expected utility theory for situations where catastrophic risks could arise. It conforms to experimental observations of how people choose under uncertainty, eliminating long standing paradoxes and experimental contradictions arising from expected utility theory [26]. In the early 1990's the introduction of 'markets with endogenous uncertainty' challenged the foundation of Arrow Debreu theory of markets under uncertainty, [17], [19], [23] [33] [22]. Endogenous uncertainty is not created by nature - it arises from the workings of the economy itself. Environmental risks are of this nature; this is a topic that Arrow studied early on [13] while formalizing irreversible decisions and introducing the notion of option values. Endogenous uncertainty views traders as participants in the creation of risk, rather than observers or victims. This is similar to Heisenberg's Uncertainty Principle in physics, where observers introduce

[1]See Edmond Malinvaud [34] [35]; later on Werner Hildenbrand followed this approach.

[2]They achieved the same for their treatement of economic dynamics. Trading over time and under conditions of uncertainty characterizes financial markets.

uncertainty in the observations. The new treatment of endogenous uncertainty breaks down the equivalence between markets with and without uncertainty that is the trademark of Arrow Debreu theory. Markets with endogenous risks cannot hedge all the risk that they create [17] and Pareto efficiency of market allocations fails except in a restricted sense [19]. Financial innovation decreases individual risk at the expense of increasing collective risks and widespread default follows from financial innovation and deregulation [22], an issue that is now observed empirically in the 2008 financial crisis. Markets with endogenous uncertainty behave quite differently from riskless markets, as Arrow has himself remarked in his recent work with Frank Hahn on the topic [12].

## 2. Brief Biographical Backgound

Kenneth Joseph Arrow, born August 23, 1921, is an American economist and joint winner of the Nobel Memorial Prize in Economics with John Hicks in 1972. Arrow is one of the founders of modern (post World War II) economic theory, and one of the most important economists of the 20th century. For a full biographical note the reader is referred to Chichilnisky [18]. Born in 1921 in New York City to Harry and Lilian Arrow, Kenneth was raised in the city. He graduated from Townsend Harris High School and earned a Bachelor's degree from the City College of New York studying under Alfred Tarski. After graduating in 1940, he went to Columbia University and after a hiatus caused by World War II, where he served with the Weather Division of the Army Air Force, he returned to Columbia University to study under the great statistician Harold Hotelling. At Columbia University, he received a Master's degree in 1941 studying under A. Wald, who was the supervisor of his Master Thesis on stochastic processes. From 1946 to 1949 he spent his time partly as a graduate student at Columbia and partly as a research associate at the Cowles Commission for Research in Economics at the University of Chicago where he met his wife Selma Schweitzer. During that time he also held the rank of Assistant Professor of Economics at the University of Chicago. Initially interested in following a career as an Actuary, in 1951 he earned his Ph.D. in Economics from Columbia University working under the supervision of Harold Hotelling and Albert Hart. His published work on risk starts in 1951 [3]. In developing his own approach to risk, Arrow grapples with the ideas of Shackle [36], Knight [31] and Keynes [30] among others, seeking and not always finding a rigorous mathematical foundation. His best known works on financial markets starts in 1953 [3] and provides a solid foundation based on the role of securities in the allocation of risks [4], [6] [5] [10], and [8]. His approach can be described as a state contingent security approach to the allocations of risks in an economy, and is largely an extension of the same approach he followed in his work on general equilibrium theory with Gerard Debreu, for which he was awarded the Nobel Prize in 1972 [7]. Nevertheless his work connects also with social issues of risk allocation and with the French literature of the time, especially Allais [1] and [2]. This article is mostly concerned with extensions of the Arrow Debreu state contingent security approach along three lines: (i) individual or idiosyncratic risk, (ii) catastrophe or rare risks and (iii) endogenous risks.

## 3. Markets under Uncertainty

Arrow Debreu theory conceptualizes uncertainty with a number of possible states of the world s =1,2,.... that may occur. Commodities can be in one of several states, and are traded separately on each of the states of nature. In this theory, one does not trade a good, but a 'contingent good', namely a good in each state of the world: apples when it rains and apples when it shines. [27] [5] [9].This way the theory of markets with $N$ goods and $S$ states of nature is formally identical to the theory of markets without uncertainty but with $N \times S$ commodities. Traders trade "state contingent commodities". This simple formulation allows one to apply the results of the theory of markets without uncertainty, to markets with uncertainty. One recovers most of the important results such as (i) the existence of a market equilibrium and (ii) the "invisible hand theorem" that establishes that market solutions are always Pareto efficient The approach is elegant, simple and general.

Yet the formulation of this theory can be unexpectedly demanding. It requires that we all agree on all the possible states of the world that describe 'collective uncertainty,' and that we trade accordingly. This turns out to be more demanding than it seems: for example, one may need to have a separate market for apples when it rains than when it does not, and separate market prices for each case. The assumption requires $N \times S$ markets to guarantee market efficiency, a requirement that in some cases militates against the applicability of the theory. In a later article Arrow simplified the demands of the theory and reduced the number of markets needed for efficiency by defining "securities" which are different payments of money exchanged among the traders in different states of nature [10]. This new approach no longer requires trading "contingent" commodities but rather trading a combination of commodities and securities. Arrow proves that by trading commodities and securities one can achieve the same results as trading state contingent commodities [10]. Rather than needing $N \times S$ markets one needs a lesser amount, $N + S - 1$ markets, namely $N$ markets for commodities and $S - 1$ markets for securities. This approach was a great improvement and led to the study of securities in a rigorous and valuable manner. The mathematical requirement to reach Pareto efficiency was simplified gradually to require that the securities traded should provide for each trader a set of choices with the same dimensionality as the original state contingent commodity approach. When this condition is not satisfied, the markets are called 'incomplete', and this led to a large literature on incomplete markets Geanakoplos ([28]) in which Pareto efficiency cannot be assured, and government intervention may be required, which exceeds the scope of this article.

## 4. Individual Risk and Insurance

Arrow - Debreu theory is not equally well suited for all types of risks. In some cases it could require an unrealistically large number of markets to reach efficient allocations. A clear example of this phenomenon arises for those risks that pertain to one individual at a time, called 'individual risks', which may not be readily interpreted as states of the world on which we all agree and are willing to trade. Individuals' accidents, illness, deaths and defaults, are frequent and important risks and fall under this category. Arrow [11] and Malinvaud [34] showed how individual uncertainty can be reformulated or reinterpreted as collective uncertainty. Malinvaud formalized the creation of states of collective risks from individual risks, by lists that describe all individuals in the economy, each in one state of individual risk. The theory of markets can be reinterpreted accordingly [34], [14] yet remains somewhat awkward. The process of trading under individual risk using Arrow Debreu theory requires an unrealistically large number of markets. For example with $N$ individuals, each in one of two individual states $G$ (good) and $B$ (bad), the number of (collective) states that are required to apply the Arrow - Debreu theory is $S = 2^N$. The number of markets required is as above either $S \times N$ or $N + S - 1$. But with $N = 300$ million people, as in the US economy, applying the Arrow Debreu approach would require $N \times S = N \times 2^{300\ million}$ markets to achieve Pareto efficiency, more markets than the total amount of particles in the known universe, see Chichilnisky and Heal [16]. For this reason individual uncertainty is best treated with another formulation of uncertainty involving individual states of uncertainty and insurance rather than securities, in which market clearing is defined on the average and may never actually occur. In this new approach, instead of requiring $N + S - 1$ markets, one requires only $N$ commodity markets and, with two states of individual risk, just one security: an insurance contract suffices to obtain asymptotic efficiency, see Malinvaud [34] [35]. This is a satisfactory theory of individual risk and insurance, but it leads only to asymptotic market clearing and Pareto efficiency. More recently, the theory was improved and it was shown that one can obtain exact market clearing solutions and Pareto efficient allocations based on $N$ commodity markets and the introduction of a limited number of financial instruments called 'mutual insurance', see Cass, Chichilnisky and Wu [14]. We establish in [14] that if there are N households (consisting of H types), each facing the possibility of being in S individual states together with T collective states, then ensuring Pareto optimality requires only H(S -1)T independent mutual insurance policies plus T pure Arrow securities .

## 5. Choice and Risk Bearing

Choice under uncertainty explains how individuals rank risky outcomes. In describing how we rank choices under uncertainty, one follows principles that were established to describe the way nature ranks what is most likely to occur, a topic that was widely explored and is at the foundation of statistics [29], [37]. To explain how individuals choose under conditions of uncertainty, Arrow used behavioral axioms that were introduced by Von Neumann and Morgenstern [38] for the theory of games[3] and axioms defined by De Groot [29] and Villegas [37] for the foundation of statistics. The main result obtained in the middle of the 20th century was that under rather simple behavioral assumptions, individuals behave as if they were optimizing an 'expected utility function'. This means that they behave as having (i) a utility $u$ for commodities that is independent from the state of nature, and (ii) subjective probabilities about how likely are the various states of nature. Using the classic axioms, from (i) and (ii) one constructs a ranking of choice under uncertainty obtaining a well known expected utility approach. Specifically, traders choose over 'lotteries' that achieve different outcomes in different states of nature. When states of nature and outcomes are represented by real numbers in $R$, a lottery is a function $f : R \to R$, a utility is a function $u : R^N \to R$ and a subjective probability is $p : R \to [0, 1]$ with $\int_R p(s) = 1$. Von Neumann, Arrow, Hernstein and Milnor, all obtained the same classic "representation theorem" that identified choice under

[3]And similar axioms used by Hernstein and Milnor [32].

uncertainty by the ranking of lotteries according to a real valued function $W$, where $W$ has the now familiar 'expected utility' form:

$$W(f) = \int_{s \in R} p(s).u(f(s))ds$$

The utility function $u$ is typically bounded to avoid paradoxical behavior.[4] The expected utility approach just described has been universally used since the mid 20-th century. Yet despite its elegance and appeal, from the very beginning expected utility has been unable to explain experimental behavior that was reported in the work of Allais [1] and others. There has been a persistent conflict between theory and observed behavior. The reason for this discrepancy has been identified more recently, and it is attributed to the fact that expected utility is dominated by frequent events and neglects rare events – even those that are potentially catastrophic, such as widespread default in today's economies. That expected utility neglects rare events was shown in Chichilnisky [20] [21] [22]. The problem was traced back to the use of Arrow's axiom of Monotone Continuity [6], which Arrow attributed to Villegas [37], and to the corresponding continuity axioms in Hernstein and Milnor and as well as De Groot [29] related continuity condition that is denoted "$SP_4$". Because of this property, the expected utility approach has been characterized as the "dictatorship" of common sense, since it is dominated by the consideration of 'normal' and frequent events. To correct this bias, and represent more accurately how we choose under uncertainty, a new axiom was added in [20] [21], requiring equal treatment for frequent and for rare events. The new axiom was subsequently proven to be the logic negation of Arrow's Monotone Continuity, an axiom that was shown to neglect small probability events [25].

The new axioms lead to a new "representation theorem" according to which the ranking of lotteries is a modified expected utility formula

$$W(f) = \int_{s \in R} p(s).u(f(s))ds + \phi(f)$$

where $\phi$ is a continuous linear function on lotteries that is best described as a finite additive measure, rather than a countably additive measure. This means a measure that assigns most weight to rare events. The new formulation has both types of measures at once, so the new characterization of choice under uncertainty incorporates both (i) frequent and (ii) rare events in a balanced manner, conforming more closely to the experimental evidence on how humans choose under uncertainty [26]. The new specification gives a well deserved importance to catastrophic risks, and a special role to fear in our decision making leading to a more realistic theory of choice under uncertainty. Yet the new theory of choice under uncertainty coincides with the old when there are no catastrophic risks so that, in reality, the latter is simply an extension of the former to incorporate rare events. Some of the most interesting applications are to environmental risks such as global warming [16]. Kenneth Arrow was an early contributor to the early literature on environmental risks and irrevesibilities [13].

[4] Specifically to avoid the so called St. Petersburg paradox, see Arrow [6].

## 6. Endogenous Uncertainty and Widespread Default

Some of the risks we face are not created by nature. They are our own creation, such as global warming or finacial crashes as those we face today in 2008. In physics the realization that the observer matters, that the observer is in reality a participant and creates uncertainty, is called Heisenberger's Uncertainty Principle. The equivalent in economics is an uncertainty principle that describes how we create risks through our economic behavior. This realization led to the new concept of 'markets with endogenous uncertainty,' created in 1991 and embodied in early articles [17] [22], [15] that established some of the basic principles and welfare theorems in markets of this sort. This and other related articles ([16] [19] [19] [22] [33] among others) established basic principles of the general equilibrium of markets with endogenous uncertainty. It is possible to extend Arrow Debreu theory of markets to encompass markets with endogenous uncertainty and also to prove the existence of market equilibrium under these conditions [19]. But in the new formulation, Heisenberg's uncertainty principle rears its quizzical face. It is shown that it no longer possible to fully hedge the risks that we create ourselves [17],. no matter how many financial instruments we create. The equivalent of Russel's paradox in mathematical logics appears also in this context due to the self - referential aspects of endogenous uncertainty [17] [19]. Pareto efficiency of equilibrium can no longer be ensured and some of the worst economic risks we face are endogenously determined. In [22] it was shown that the creation of financial instruments to hedge individual risks - such as credit default insurance that are often a subject of discussion in today's financial turmoil - by themselves induce collective risks of widespread default. The widespread default that we experience today was anticipated in [22] in 1991 and 2006, where it was attributed to endogenous uncertainty created by financial innovation and our choices of regulation or deregulation of financial instruments. Examples are the extent of reserves that are required for investment banking operations, and the creation of mortgage - backed securities that are behind many of the default risks that we face today [23]. Financial innovation of this nature and the attendant regulation of new financial instruments, causes welfare gains for individuals but at the same time create risks of welfare losses for society that bears the collective risks that ensue, as is observed in 2008. In this context Arrow Debreu theory of markets can no longer treat markets with endogenous uncertainty as it treats markets with standard commodities. The symmetry of markets with and without uncertainty is now broken. We face a brave new world of financial innovation and the endogenous uncertainty that we create ourselves. Creation and hedging of risks are closely linked, and endogenous uncertainty has acquired a critical role in market performance and economic welfare, an issue that Kenneth Arrow has tackled himself through joint work with Frank Hahn in this literature [12].

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