Catastrophical risk

Graciela Chichilnisky

Volume 1, pp 274–279

in

Encyclopedia of Environmetrics
(ISBN 0471 899976)

Edited by

Abdel H. El-Shaarawi and Walter W. Piegorsch

Catastrophical risk

Global environmental risks such as climate change and rising sea levels are low-probability events with widespread and possibly irreversible consequences. These are fundamentally new risks which are not well understood. Learning through experimentation is out of the question because these risks are effectively irreversible in a time-scale that matters. As a result, classical theories that rely on expected utility (*see* **Utility theory**) may not work well because they underestimate low-probability events, as discussed below. The need to make global environmental decisions calls for a systematic analysis of choices involving low-probability events with major irreversible consequences. The topic is of current importance but has been neglected in the literature of choice under uncertainty.

This entry introduces a new decision-making tool for such situations. First, it shows why the classical Von Neumann axioms do not work well in this context, as they lead to expected utility that can be insensitive towards small-probability events. Secondly, the entry introduces and develops a new set of axioms that require sensitivity to both small- and large-probability events. These axioms appear to represent ways in which people rationalize the problem of making decisions in situations involving catastrophical risks. The axioms are different from the classic axioms by Von Neumann and Morgenstern, and they lead to a different decision theory which is not based on expected utility analysis. Finally, through a representation theorem, it is shown that all the criteria implied by the new axioms have the following form: one term that takes into account the maximization of expected utility, plus a second term which is a well-defined operator that can be interpreted as a desire to avoid a catastrophe. Both parts are present, and both turn out to be important in making decisions under catastrophical risks. This entry provides practical examples of how to use these criteria. It shows how the new axioms help to explain the Allais paradox (see below), which involves choices with low-probability events, and suggests new questions on game theory and on the calculus of variations.

Von Neumann–Morgenstern Axioms

A set of mathematical axioms introduced half a century ago by John Von Neumann and Oscar Morgenstern gave rise to a now classical tool for decision making under uncertainty. Several other mathematicians and economists, such as Hernstein, Milnor and Arrow, developed related axioms [7]. The axioms formalize the properties of orders defined on sets of uncertain events; the orders are then used to rank or evaluate risky outcomes. The structure of the decision problem is simple. A system with uncertain characteristics is in one of several possible states; each state is the value of a random variable which describes the system. For example: the average temperature of the planet's surface is a state. The system's states can be described by real numbers. For each state $s \in R$ there is an associated outcome; for example, for each temperature level there is an associated vector describing soil fertility and precipitation. Therefore one has $x(s) \in R^N$, $N \geq 1$. When the probabilities associated with each state are given, a description of outcomes across all states is called a *lottery*. (A lottery is also described by the probabilities of each state and the outcomes in each state.) A lottery is therefore a function $x : R \to R^N$, and the space of all lotteries is a function space L.

The Von Neumann–Morgenstern (NM) axioms provide a mathematical formalization of how to rank or order lotteries, i.e. of reasonable ways to order the elements of L. The NM model presumes that the outcomes themselves are ranked; it creates a utility index for the outcomes that is consistent with this ranking and a decision criterion for choice among lotteries. In this sense the NM model does two jobs at once. **Optimization** according to such an order defines a form of decision making under uncertainty used widely until now.

A main result obtained from the NM axioms is a representation theorem: a characterization of all the functionals on L which satisfy the NM axioms. Maximizing such a functional $W : L \to R$ over a constrained set given by initial conditions defines a form of rational choice under uncertainty. Von Neumann and Morgenstern proved that an order over lotteries which satisfies their axioms admits a representation by an integral operator $W : L \to R$, which has as a kernel a countably additive measure over the set of states. Such operators are called Von Neumann–Morgenstern utilities, and the decision procedure obtained by optimizing such utilities is called expected utility maximization, so that

$$W(x) = \int_{s \in R} u(x(s))\, \mathrm{d}\mu(s) \qquad (1)$$

where the real line R is the state–space; the function $x : R \to R^N$ is a lottery; $u : R^N \to R$ is a utility function describing the utility provided by the outcome of the lottery in each state $s, u(s)$; and $\mathrm{d}\mu(x)$ is a countably additive measure that defines a probability distribution over measurable subsets of states in R. It is standard practice to require that the utility function is bounded to avoid the St Petersburg paradox [2, Chapter 3]. The assumption of *bounded utility* is sufficient but not necessary to avoid the St Petersburg paradox. As Bernoulli pointed out, the logarithmic utility function $u(x) = \log(x)$ would deliver a finite expected utility in the context of the St Petersburg paradox. However, we need $p(s)u(x(s))$ to converge to zero as $s \to \infty$ for the integral to exist. According to the NM representation theorem, a rational choice under uncertainty which satisfies the NM axioms must take the following form: a lottery x is ranked above another y if and only if W assigns to x a larger real number. In symbols

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where W satisfies (1).

The optimization of expected utility is a widely used procedure for evaluating choices under uncertainty. Mathematically, functionals such as W are convenient because they are amenable to a large body of knowledge which goes back several centuries: the calculus of variations. The Euler–Lagrange equations are typically used to characterize optimal solutions. Such mathematical tools are widely used and are valuable to find and describe choices under uncertainty.

Catastrophical Risks

A catastrophical risk is a low-probability event which can lead to major and typically irreversible losses. As already mentioned, global environmental problems have these characteristics (*see* **Global environmental change**). The classical methods defined above, despite their widespread use, are not satisfactory to evaluate catastrophical risks. The reasons are both practical and theoretical. From the practical point of view, it has been shown that using such criteria undervalues catastrophical risks and hence conflicts with the observed evidence of how humans evaluate such risks. For example using NM utilities, the most damaging scenarios of **global warming** induce little if any economic loss. The Intergovernmental Panel on Climate Change (IPCC), the main international scientific organization in this area, recently announced a highly contested figure of about 2% loss of economic value from a doubling of CO_2 concentration in the atmosphere. This is a symptom of a more general phenomenon; a simple computation shows that the hypothetical disappearance of all irrigation water in the US and all the country's agricultural produce would have at most a 2.5% impact on its gross domestic product. This finding underscores the importance of using appropriate criteria to evaluate catastrophical risks.

Mathematically the problem arises from the fact that the expected utility operator W which emerges from the NM representation theorem (1) is defined with respect to a probability measure μ, which is therefore countably additive. Since the utility function $u : R^N \to R$ is bounded (i.e. $\sup_{x \in R} |u(x)| < \infty$), the countable additivity of μ can be shown to imply that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$, where ε depends on x and y. To show this formally, introduce the following definition.

Definition 1 A functional $W : L \to R$ is said to be insensitive to small probability events when

$$\begin{aligned} W(x) > W(y) &\Leftrightarrow \exists \varepsilon > 0 \\ W(x') > W(y') & \end{aligned} \tag{2}$$

for all x', y' such that

$$x' = x \quad \text{and} \quad y' = y, \quad \text{a.e. on any set}$$
$$A : A^c \subset R : \mu(A) < \varepsilon$$

The interpretation of this definition is that W ranks x above y if and only if it ranks x' above y' for any pair of lotteries x' and y' which are obtained by modifying arbitrarily x and y in sets of states within a set A with probability lower than ε. Under these conditions one says that the ranking defined by W is *insensitive* to the outcomes of the lottery in small probability events. The following lemma shows that, as defined by NM, the expected utility criterion W is not well-suited to evaluate catastrophical risks. For simplicity of notation, and without loss of generality, let $N = 1$; the same results hold for arbitrary N.

Lemma *Expected utility is insensitive to catastrophical risks.*

Proof The expected utility criterion ranks lotteries in L as follows: $x(s) \succ y(s) \Leftrightarrow \exists$ a measurable and bounded utility function $u : R \to R$, and a probability measure μ on R:

$$\int_R u(x(s))\,\mathrm{d}\mu(s) \succ \int_R u(y(s))\,\mathrm{d}\mu(s)$$

Now

$$\int_R u(x(s))\,\mathrm{d}\mu(s) \succ \int_R u(y(s))\,\mathrm{d}\mu(s) \Leftrightarrow \exists\delta > 0$$
$$\int_R u(x(s))\,\mathrm{d}\mu(s) \succ \int_R u(y(s))\,\mathrm{d}\mu(s) + \delta$$

Let

$$\varepsilon = \varepsilon(x, y) = \frac{\delta}{6K} \tag{3}$$

where

$$K = \sup_{x\in L, s\in R} |u(x(s))| \tag{4}$$

If

$$x' = x \quad \text{and} \quad y' = y, \quad \text{a.e. on } S^c \tag{5}$$

where $\mu(S) < \varepsilon$, then

$$\left|\int_R u(x(s))\,\mathrm{d}\mu(s) - \int_R u(x'(s))\,\mathrm{d}\mu(s)\right| < 2K\mu(S) < \frac{\delta}{3}$$

and

$$\left|\int_R u(y(s))\,\mathrm{d}\mu(s) - \int_R u(y'(s))\,\mathrm{d}\mu(s)\right| < 2K\mu(S) < \frac{\delta}{3}$$

Therefore

$$x \succ y \Rightarrow \int_R u(x'(s))\,\mathrm{d}\mu(s) > \int_R u(y'(s))\,\mathrm{d}\mu(s) \Rightarrow x' \succ y'$$

Reciprocally

$$x' \succ y' \Rightarrow x \succ y$$

So that for $\varepsilon = \delta/6K$

$$x \succ y \Leftrightarrow \exists\varepsilon > 0 : x' \succ y' \quad \text{when} \quad x = x' \quad \text{and} \quad y = y' \quad \text{a.e. on any } S : \mu(S^c) < \varepsilon$$

and therefore by definition the expected utility criterion is insensitive to small probability events.

By the result just established, cost–benefit analysis under uncertainty based on expected utility maximization underestimates the outcomes of small-probability events. It is thus biased against certain environmental projects that are designed to prevent catastrophical events. Experimental evidence shows that humans treat choices under uncertainty somewhat differently from what the NM axioms would predict (*see* **Risk perception**), and it raises questions about the need for alternative axioms which describe more accurately human beings' valuations.

Updating NM

Recently a new set of axioms has been developed which update the NM axioms to correct the bias mentioned against small-probability events. A well-defined set of axioms which contrast with the NM axioms was introduced in [4], along with the attendant representation theorems, identifying new types of functionals which are maximized under uncertainty. These axioms parallel similar axioms and criteria for choice over time introduced in [5] and [6]. (See also [8] for an alternative analysis to the NM treatment of decision making under uncertainty that does not provide an axiomatic treatment.)

New Axioms for Choice Under Uncertainty

We propose three axioms for choice under uncertainty, which must be satisfied by the criterion $W : L \to R$ that is used to evaluate lotteries. The first axiom is satisfied by the expected utility; the other two are not. The first axiom involves linearity and continuity of the criterion with respect to the utility derived from lotteries, where continuity is defined with respect to the sup norm on the space of utility values associated with lotteries L. Formally, the utility values of lotteries L are in the space of measurable and essentially bounded functions on R, with the norm $\|u(x(s))\| = \sup_{x\in L, s\in R} |u(x(s))|$.

Axiom 1: Continuity of the functional W with respect to its argument, the utility of the lottery $u(x)$.

Axiom 2: Sensitivity to low-probability events. This rules out insensitivity to low-probability events as in the above definition and lemma.

Axiom 3: Sensitivity to large-probability events. This rules out insensitivity to events of large probability, as defined below.

Definition 2 A ranking is said to be insensitive to large-probability events when $\forall x, y\, \exists \varepsilon > 0, \varepsilon(x, y)$ such that

$$W(x) > W(y) \Leftrightarrow W(x') \geq W(y') \qquad (6)$$

for all lotteries x', y' such that $x = x', y = y'$, a.e. on S^c, where $\mu(S) > 1 - \varepsilon$. In words: the ranking is the same on any two lotteries x' and y' that are obtained by modifying arbitrarily x and y in any bounded set of states $S \subset R$, which may have an arbitrarily large probability.

Example 1 As an example of a function which is insensitive to large-probability events, consider the space of all continuous linear real-valued functions on L_∞, the dual of L_∞, denoted L_∞^*. Within this dual consider a purely finitely additive measure ν on R which assigns measure zero to any bounded set in R, i.e. $\nu(S) = 0$ if $\forall_x \in S, |x| < K$, for some $K > 0$. Such measures define functionals satisfying (6). Such functionals are ruled out by Axiom 3, which requires sensitivity to large-probability events. Indeed, such functionals put all the weight on infinity, i.e. on events of arbitrarily small probabilities according to the countably additive measure μ on R.

A Representation Theorem

Like the NM axioms, the three new axioms presented above lead to a representation theorem establishing the form of every ranking of lotteries that satisfies the three axioms given above. It has been shown [4] that there exist functionals $\Psi : L_\infty \to R$ which rank all lotteries and satisfy all the axioms. Rather than countably additive kernels, however, these functionals are a convex combination of integral operators with countably additive kernels and purely finitely additive measures, with both elements (countably and finitely additive) nonzero.

Theorem *Any ranking $\succ$ of lotteries in $L = L_\infty(R)$ satisfying the three axioms defined above must be of the form*

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where $W : L \to R$

$$W(x) = \lambda \left[\int_R u[x(s)]\, \mathrm{d}\mu(s) \right] + (1 - \lambda)\Phi[u(x(s))] \qquad (7)$$

for $\lambda \in (0, 1)$, $u : R \to R$, μ a probability measure on R, and $\Phi : L \to R, \Phi \in L^* - L_1$ is a purely finitely additive measure.

Proof The proof follows the line of argument presented in [5] and [6]. As defined above, the space of all utility functions derived from lotteries is $L_\infty(R)$ with the sup norm. By Axiom 1, we are looking for an element of the dual space $L_\infty^*(R)$, the space of all continuous linear real-valued functions on $L_\infty(R)$. By standard results in functional analysis, the dual space $L_\infty^*(R)$ consists of $L_1(R)$ as well as another space consisting of purely finitely additive measures, namely continuous linear functions that assign the value zero to any function supported on a bounded set of R. By Axiom 2, the function W is not contained in L_1, since in that case as shown in the above lemma, Axiom 2 is violated. Axiom 3 implies that W is not a purely finitely additive measure either; as shown in [5] and [6] the only possible form is as represented above.

Remark The connection between the function $W(x)$ and the Prospect theory of Kahneman and Tversky is the subject of another entry.

Example 2 As an illustration of the representation theorem presented above, consider the case when the states are discrete, indexed by the integers Z. For each real number μ, $0 < \mu < 1$, a continuous linear functional $\Psi : l_\infty \to R$ can be defined as follows:

$$\Psi(x) = \mu \sum_{s=1}^{\infty} \lambda^{-s} u(x(s)) + (1 - \mu) \lim_{s \in Z} u(x(s)) \quad (8)$$

where $\lim_{s\in Z} u(x(s))$ is the (Hahn–Banach) extension of the continuous linear limit operator to the space l_∞ of all bounded real-valued functions on Z. The interpretation of the two parts of the function Ψ in (4) is simple. The first part is an integral operator with an integrable kernel $\{\lambda^{-s}\}_{s\in Z}$ which defines a countably additive measure on Z, and therefore emphasizes the weight of large-probability events in the ranking of a lottery $x \in l_\infty$. The second part defines a purely finitely additive measure on Z which assigns positive weight to small-probability events. It defines a measure with 'heavy tails'. Both parts are present, so Ψ is sensitive to small- and large-probability events. Catastrophical risks are therefore ranked more realistically by such functionals. The mathematics involved in these representation results is nonlinear analysis, as well as the analysis of convex systems.

Examples and Open Questions

Examples

Consider an electrical utility company that seeks. They seek to implement a production and service plan which would be optimal under normal conditions, while at the same time avoiding a potentially catastrophical black-out incident (which could be costly in monetary terms and in human lives). Following our axioms and the above theorem, a typical criterion that would be adopted would involve choosing among all possible plans to maximize the expected throughput plus minimizing the probability of reaching a critical level beyond which there would be a black-out. It can be shown that such a criterion would satisfy our three axioms.

The Allais Paradox

The first and perhaps most famous violation of the standard models of choice under uncertainty is due to M. Allais, who presented experimental evidence which is inconsistent with the NM axioms. A variation of this paradox was reported by Kahneman and Tversky. They observed that 82% of the subjects chose a gamble A over another gamble B, and 83% of the subjects chose a gamble C over another D, so that at least 65% chose B and C. However, as shown below, this pair of gambles B and C is inconsistent with the NM model of expected utility.

Example Gamble A consists of a 0.33 chance of winning \$2500, a 0.66 chance of winning \$2400, and a 0.01 chance of winning \$0; while gamble B is a 1.0 chance of winning \$2400. Gamble C consists of a 0.33 chance of winning \$2500, and a 0.67 chance of winning \$0; while gamble D consists of a 0.34 chance of winning \$2400 and a 0.66 chance of winning \$0.

Observe that if an individual prefers B over A, this means that their (sure) utility function u over income satisfies

$$u(2400) > 0.33u(2500) + 0.66u(2400) + 0.01u(0)$$

or

$$0.34u(2400) > 0.33u(2500) + 0.01u(0) \qquad (9)$$

the latter of which contradicts a choice in favor of C, because choosing C over D implies

$$0.33u(2500) + 0.67u(0) > 0.34u(2400) + 0.66u(0) \qquad (10)$$

One way to resolve this paradox is to understand that, when the new axioms are taken into consideration, the individual's utility function u has two components in cases of small-probability events: one of these components is the expected utility, and the other is focused on the small-probability event $u(0)$. Therefore (9) above can be now written as

$$0.34u(2400) > 0.33u(2500) + 0.01u(0) - \theta$$

for some real number $\theta > 0$, representing a higher weight given to the low (0.01) probability event of winning \$0 than would be the case with expected utility. This implies that

$$0.33u(2500) + 0.67u(0) - \theta < 0.34u(2400) + 0.66u(0)$$

which is no longer inconsistent with (10). With the new axioms, therefore, (9) no longer contradicts (10) and the Allain paradox has been resolved.

Open Questions

Risk aversion is typically defined with respect to the utility function which appears inside the expected utility functional (1). Here this definition may not work, and an alternative definition may be needed. An interesting open question is how to define risk aversion for the functionals in (1), which satisfy our axioms.

Another question is how to define repeated game solutions (e.g. Nash equilibrium) that involve players with welfare functions of the forms identified here, and to explore when these solutions exist.

The traditional calculus of variation is based on integral operators that have finite kernels, such as exponential weight functions of the form $e^{-\nu s}$. This specification no longer holds here, and therefore the optimization of the operators emerging from the new axioms require a new form of calculus of variation. It is of interest that standard tools of the calculus of variations must be redeveloped in new directions. Some results already exist [5, 6], but much work is still needed. The study of optimal solutions of this type of functional has led to asymptotically autonomous dynamical systems, which occur naturally when one extends the Euler–Lagrange analysis of optimal solutions to encompass the type of operators defined here. Statistical analysis of such systems also requires new tools.

References

[1] Allais, M. (1988). The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function, in *Risk, Decision and Rationality*, B.R. Munier, ed., Reidel, Dordrecht, pp. 233–289.

[2] Arrow, K. (1971). *Essays in the Theory of Risk Bearing*, North-Holland, Amsterdam.

[3] Cass, D., Chichilnisky, G. & Wu, H. (1996). Individual risks and mutual insurance, *Econometrica* **64**, 333–341.

[4] Chichilnisky, G. (1996). Updating Von Neumann Morgenstern axioms for choice under catastrophic risks, *Workshop on Catastrophic Environmental Risks*, The Fields Institute of Mathematical Studies, University of Toronto, Canada.

[5] Chichilnisky, G. (1996). An axiomatic approach to sustainable development, *Society of Choice and Welfare* **13**, 321–357.

[6] Chichilnisky, G. (1997). What is sustainable development?, *Land Economics* **73**, 467–491.

[7] Hernstein, N. & Milnor, J. (1953). An axiomatic approach to measurable utility, *Econometrica* **21**, 291–297.

[8] Machina, M. (1982). Expected utility analysis without the independent axiom, *Econometrica* **50**, 277–323.

[9] Machina, M. (1989). Dynamic consistency and non-expected utility models of choice under uncertainty, *Journal of Economic Literature* **XXii**, 1622–1668.

[10] Tversky, A. & Wakker, P. (1995). Risk attitudes and decision weights, *Econometrica* **63**, 1225–1280.

(*See also* **Decision support systems**; **Economics, environmental**; **Sustainability**)

GRACIELA CHICHILNISKY