ELSEVIER Resource and Energy Economics 22 (2000) 221–231

RESOURCE and ENERGY ECONOMICS

www.elsevier.nl/locate/ECONbase

# An axiomatic approach to choice under uncertainty with catastrophic risks

Graciela Chichilnisky *,[1]

*Department of Statistics, Mathematics Building, Columbia University, New York, NY 10027, USA*

Received 14 June 1998; accepted 2 November 1998

The Fields Institute for Mathematical Sciences, Toronto, June 9–11, 1996

**Abstract**

This paper analyses decision under uncertainty with catastrophic risks, and is motivated by problems emerging from global environmental risks. These are typically low-probability events with major irreversible consequences. For such risks, the Von Neumann–Morgenstern (NM) axioms for decision making under uncertainty are not appropriate, since they are shown here to be insensitive to low-probability events. The paper introduces an alternative set of axioms requiring sensitivity to both low- and large-probability events. Through a new representation theorem in functional analysis, the results characterize all the operators whose maximization leads to the fulfillment of these axioms. They involve a convex combination of expected utility and a criterion based on the desire to avoid low probability and potentially catastrophic events. It is shown that the new axioms help resolve the Allais paradox. Open questions about risk aversion, games under uncertainty and calculus of variations are discussed. © 2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Von Neumann–Morgenstern axioms; Allais paradox; Catastrophic events; Decision theory; Uncertainty

* Tel.: +1-212-678-0097; fax: +1-212-678-0405.

*E-mail address:* gc9@columbia.edu (G. Chichilnisky).

[1] UNESCO Professor of Mathematics and Economics, Professor of Statistics and Director, Center for Risk Management, Columbia University, New York, NY 10027

0928-7655/00/$ - see front matter 

PII: S0928-7655(00)00032-4

## 1. Introduction

Global environmental risks such as climate change and rising sea levels are low-probability events with major, widespread (cf. Cass et al., 1996) and possibly irreversible consequences. These are fundamentally new risks that are not well understood. Learning through experimentation is out of the question because these risks are effectively irreversible in a timescale that matters. As a result, classic theories, which rely on expected utility, may not work well because they underestimate low-probability events, as proven below. The need to make global environmental decisions calls for a systematic analysis of choices involving low-probability events with major irreversible consequences. The topic is of current importance but has been neglected in the literature of choice under uncertainty.

This paper introduces a new decision making tool for such situations. Firstly, it shows why the classic Von Neumann axioms do not work well in this context, as they lead to expected utility criteria that can be insensitive towards small-probability events. Secondly, the paper introduces and develops a new set of axioms requiring sensitivity to both small and large-probability events. These axioms appear to represent ways in which people rationalize the problem of making decisions in situations involving catastrophic risks. The axioms are different from the classic axioms by Von Neumann and Morgenstern and lead to a different decision theory, which is not based on expected utility analysis. Finally, through a representation theorem, I show that all the criteria implied by the new axioms have the following form: one term that takes into account the maximization of expected utility plus a second term, which is a well-defined operator that can be interpreted as a desire to avoid a catastrophe. Both parts are present, and both turn out to be important in making decisions under catastrophic risks. The paper provides practical examples of how to use these criteria. It shows how the new axioms help explain the so-called Allais paradox (Allais, 1988), which involves choices with low-probability events and suggests new questions on game theory and on the calculus of variations.

## 2. Von Neumann–Morgenstern (NM) axioms

A set of mathematical axioms introduced half a century ago by John Von Neumann and Oscar Morgenstern gave rise to a now classical tool for decision making under uncertainty. Several other mathematicians and economists, such as Hernstein, Milnor and Arrow, developed related axioms (Hernstein and Milnor, 1953). The axioms formalize the properties of orders defined on sets of uncertain events; the orders are then used to rank or evaluate risky outcomes. The structure of the decision problem is simple. A system with uncertain characteristics is in one of several possible states; each state is the value of a random variable, which describes the system. For example: the average temperature of the planet's surface is a state. The system's states can be described by real numbers.

To each state $s \in R$, there is an associated outcome, for example to each temperature level, there is an associated vector describing soil fertility and precipitation. Therefore, one has $x(s) \in R^N$, $N \geq 1$. When the probabilities associated to each state are given, a description of outcomes across all states is called a "lottery".[2] A lottery is therefore a function $x: R \to R^N$, and the space of all lotteries is a function space $L$.

The NM axioms provide a mathematical formalization of how to rank or order lotteries, i.e. of what reasonable ways to order the elements of $L$.[3] Optimization according to such an order defines standard decision making under uncertainty.

A main result obtained from the NM axioms is a representation theorem: a characterization of all the functionals on $L$, which satisfy the NM axioms. Maximizing such a functional $W: L \to R$ over a constrained set given by initial conditions defines rational choice under uncertainty. NM proved that an order over lotteries, which satisfies their axioms, admits a representation by an integral operator $W: L \to R$, which has, as a kernel, a countably additive measure over the set of states. Such operators are called "NM utilities" and the decision procedure obtained by optimizing such utilities is called "expected utility maximization", so that:

$$W(x) = \int_{s \in R} u(x(s)) \mathrm{d}\mu(s), \qquad (1)$$

where the real line $R$ is the state space, the function $x: R \to R^N$ is a "lottery", $u: R^N \to R$ is a "utility function" describing the utility provided by the outcome of the lottery in each state $s$, $u(s)$, and $\mathrm{d}\mu(x)$ is a countably additive measure defining a probability distribution over measurable subsets of states in $R$. It is standard to require that the utility function is bounded to avoid the St. Petersburg paradox[4] (Arrow, 1971, Chap. 3) so that the utility values of all lotteries in $L$ are uniformly bounded. According to the NM representation theorem, rational choice under uncertainty that satisfies the NM axioms, must take the following form: a lottery $x$ is ranked above another $y$ if and only if $W$ assigns to $x$ a larger real number. In symbols:

$$x \succ y \Leftrightarrow W(x) > W(y),$$

where $W$ satisfies Eq. (1).

[2] A lottery is also described by the probabilities of each state and the outcomes in each state.

[3] The NM model presumes that the outcomes themselves are ranked: it creates a utility index for the outcomes consistent with this ranking and a decision criterion for choice amongst lotteries. In this sense, the NM model does two jobs at once.

[4] The assumption of "bounded utility" is sufficient but not necessary for avoiding the St. Petersburg paradox: As Bernoulli pointed out, the logarithmic utility function $u(x) = \log(x)$, would deliver a finite expected utility in the context of the St. Petersburg paradox. However, we need $p(s)u(x(s))$ to converge to zero as $s \to \infty$ for the integral to exist.

The optimization of expected utility is a widely used procedure for evaluating choices under uncertainty. Mathematically, functionals such as $W$ are convenient because they are amenable to a large body of knowledge that goes back several centuries: the calculus of variations. The Euler Lagrange equations are typically used to characterize optimal solutions. Such mathematical tools are widely used and very valuable to find and describe choices under uncertainty.

## 3. Catastrophic risks

A catastrophic risk is a low-probability event, which can lead to major and typically irreversible losses. As already mentioned, global environmental problems have these characteristics. The classic methods defined above, despite their widespread use are not satisfactory for evaluating catastrophic risks. The reasons are both practical and theoretical. From the practical point of view, it has been shown that using such criteria undervalues catastrophic risks and hence conflicts with the observed evidence of how humans evaluate such risks. For example using NM utilities, the most damaging scenarios of global climate change induce little if any economic loss. The Intergovernmental Panel on Climate Change (IPCC), the main international scientific organization in this area, recently predicted a highly contested figure of about 2% loss of economic value from a doubling of $CO_2$ concentration in the atmosphere. This is a symptom of a more general phenomenon: a simple computation shows that the hypothetical disappearance of all irrigation water in the USA and all the country's agricultural produce would have at most a 2 1/2% impact on its gross domestic product. This finding underscores the importance of using appropriate criteria for evaluating catastrophic risks.

Mathematically, the problem arises from the fact that the expected utility operator $W$ that emerges from the NM representation theorem (1) is defined with respect to a probability measure $\mu$, which is therefore a countably additive. Since the ''utility'' function $u: R^N \to R$ is bounded (i.e. $\sup_{x \in R}|u(x)| < \infty$), the countable additivity of $\mu$ can be shown to imply that any two lotteries $x, y \in L$ are ranked by $W$ quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$, where $\varepsilon$ depends on $x$ and $y$. To show this formally, I introduce the following definition:

**Definition 1.** A functional $W: L \to R$ is called ''insensitive to small-probability events'' when:

$$W(x) > W(y) \Leftrightarrow \exists \varepsilon > 0:$$

$$W(x') > W(y') \tag{2}$$

for all $x'$, $y'$ such that

$$x' = x \text{ and } y' = y \text{ a.e. on } A^c \subset R: \mu(A) < \varepsilon.$$

The interpretation of this definition is that $W$ ranks $x$ above $y$ if and only if it ranks $x'$ above $y'$ for any pair of lotteries $x'$ and $y'$ that are obtained by modifying arbitrarily $x$ and $y$ in sets of states $A$ with probability lower than $\varepsilon$. Under these conditions, one says that the ranking defined by $W$ is ''insensitive'' to the outcomes of the lottery in small-probability events. The following lemma shows that, as defined by NM, the expected utility criterion $W$ is not well suited for evaluating catastrophic risks. For simplicity of notation and without loss of generality, let $N=1$; the same results hold for arbitrary $N$.

**Lemma 1.** *Expected utility is insensitive to small-probability events.*

**Proof.** The expected utility criterion ranks lotteries in $L$ as follows: $x(s) \succ y(s) \Leftrightarrow \exists$ a measurable and bounded utility function $u: R \to R$ and a probability measure $\mu$ on $R$ such that:

$$\int_R u(x(s))\mathrm{d}\mu(s) \succ \int_R u(y(s))\mathrm{d}\mu(s).$$

Now

$$\int_R u(x(s))\mathrm{d}\mu(s) \succ \int_R u(y(s))\mathrm{d}\mu(s) \Leftrightarrow \exists \delta > 0:$$

$$\int_R u(x(s))\mathrm{d}\mu(s) \succ \int_R u(y(s))\mathrm{d}\mu(s) + \delta.$$

Let

$$\epsilon = \epsilon(x,y) = \delta/6K,$$

where

$$K = \mathrm{Sup}_{x \in L,\, s \in R} |u(x(s))|.$$

If

$$x' = x \text{ and } y' = y \text{ a.e. on } S^c,$$

where

$$\mu(S) < \epsilon,$$

then

$$\left|\int_R u(x(s))\mathrm{d}\mu(s) - \int_R u(x'(s))\mathrm{d}\mu(s)\right| < 2K\mu(S) < \delta/3,$$

and

$$\left|\int_R u(y(s))\mathrm{d}\mu(s) - \int_R u(y'(s))\mathrm{d}\mu(s)\right| < 2K\mu(S) < \delta/3.$$

Therefore,

$$x \succ y \Rightarrow \int_R u(x'(s))\mathrm{d}\mu(s) > \int_R u(y'(s))\mathrm{d}\mu(s) \Rightarrow x' \succ y'.$$

Reciprocally:

$$x' \succ y' \Rightarrow x \succ y,$$

so that for $\epsilon = \delta/6K$

$$x \succ y \Leftrightarrow \exists \epsilon > 0: x' \succ y' \text{ when } x = x' \text{ and } y = y' \text{ a.e. on any } S: \mu(S^c) < \epsilon$$

and therefore by definition, the expected utility criterion is insensitive to small-probability events. □

By the result just established, cost–benefit analysis under uncertainty based on expected utility maximization underestimates the outcomes of small-probability events. It is biased against certain environmental projects that are designed to prevent catastrophic events. Experimental evidence shows that humans treat choices under uncertainty somewhat differently from what the NM axioms would predict and raises questions about the need for alternative axioms, which describe more accurately human beings' valuations.

## 4. Updating NM axioms

Recently, a new set of axioms, which update NM axioms to correct the bias pointed out in Section 3 against small-probability events, has been developed. Chichilnisky (1996a) introduced a well-defined set of axioms that contrast with NM axioms and produced the attendant representation theorems, identifying new types of functionals that are maximized under uncertainty. These axioms parallel similar ∀ axioms and criterion for choice over time introduced in Chichilnisky (1996b, 1997).[5]

## 5. New axioms for choice under uncertainty

We propose three axioms for choice under uncertainty that must be satisfied by the criterion $W: L \to R$ used to evaluate lotteries. The first axiom is satisfied by expected utility; the other two are not. The first axiom involves linearity and continuity of the criterion with respect to the utility derived from lotteries, where continuity is defined with respect to the sup norm on the space of utility values associated with lotteries $L$. Formally, utility values of lotteries $L$ are in the space of measurable and essentially bounded functions on $R$, with the norm $\|u(x(s))\| = \text{Sup}_{x \in L,\, s \in R} |u(x(s))|$.

[5] See also Machina (1982, 1989) for an alternative analysis to NM treatment of decision making under uncertainty. Machina does not provide an axiomatic treatment.

**Axiom 1.** Continuity of the functional $W$ with respect to its argument, the utility of the lottery $u(x)$.[6]

**Axiom 2.** Sensitivity to low-probability events. This rules insensitivity to low-probability events as in Definition 1 above.

**Axiom 3.** Sensitivity to large-probability events. This rules out insensitivity to events of large probability, as defined below:

**Definition 2.** A ranking is said to be insensitive to large-probability events when $\forall x,\ y \exists \epsilon > 0,\ \epsilon\ (x,y)$ such that

$$W(x) > W(y) \Leftrightarrow W(x') \geq W(y') \tag{3}$$

for all lotteries $x'$, $y'$ such that $x = x'$, $y = y'$ a.e. on $S^c$ where $\mu(S) > 1 - \epsilon$. In words: the ranking is the same on any two lotteries $x'$ and $y'$ that are obtained by modifying arbitrarily $x$ and $y$ in any bounded set of states $S \subset R$, which may have an arbitrarily large probability.

**Example 1.** As an example of a function that is insensitive to large-probability events, consider the space of all continuous linear real valued functions on $L_\infty$, the ''dual'' of $L_\infty$, denoted $L_\infty^*$. Within this dual, consider a ''purely finitely additive measure'' $\nu$ on $R$ that assigns measure zero to any bounded set in $R$, i.e. $\nu(S) = 0$ if $\forall x \in S,\ |x| < K$, for some $K > 0$. Such measures define functionals satisfying Eq. (3). Such functionals are ruled out by Axiom 3, which requires sensitivity to large-probability events. Indeed, such functionals put all the ''weight'' on infinity, i.e. on events of arbitrarily small probabilities according to the countably additive measure $\mu$ on $R$.

## 6. A representation theorem

Like the NM axioms, the three new axioms presented above lead to a representation theorem establishing the form of every ranking of lotteries that satisfies the three axioms given above. It has been shown in Chichilnisky (1992),[7] that there exist functionals $\Psi : L_\infty \to R$ that rank all lotteries and satisfy all the axioms. Rather than countably additive kernels, however, these functionals are a convex combination of integral operators with countably additive kernels and

[6] Continuity is defined with respect to the sup norm on the space of utility values associated with lotteries $L$. The space of utility values of lotteries $L$ is the space of all measurable essentially bounded functions on $R$, and the sup norm is defined as $\|u(x(s))\| = \mathrm{Sup}_{x \in L,\ s \in R} |u(x(s))|$.

[7] Op. cit.

purely finitely additive measures, with both elements (countably and finitely additive) non-zero.

**Theorem 1.** *Any ranking $\succ$ of lotteries in $L = L_\infty(R)$ satisfying the three axioms defined must be of the form:*

$$x \succ y \Leftrightarrow W(x) > W(y),$$

*where $W: L \to R$*

$$W(x) = \lambda\left[\int_R u(x(s))\,\mathrm{d}\mu(s)\right] + (1-\lambda)\Phi(u(x(s))),$$

*for $\lambda \in (0,1)$, $\mu: R \to R$, $\mu$ is a probability measure on $R$, and $\Phi: L \to R$, $\Phi \in L^* - L_1$ is a purely finitely additive measure.*

**Proof.** The proof follows the line of argument presented in Chichilnisky (1996b, 1997). As defined above, for any bounded utility $\mu$, the space of all utility values derived from lotteries can be described as a bounded subset in $L_\infty(R)$ with the sup norm. By Axiom 1, we are looking for an element of the dual space $L_\infty^*(R)$, the space of all continuous linear real valued functions on $L_\infty(R)$. By standard results in functional analysis, the dual space $L_\infty^*(R)$ consists of $L_1(R)$ as well as another space consisting of ''purely finitely additive'' measures, namely continuous linear functions that assign zero value to any function supported on a bounded set of $R$. By Axiom 2, the function $W$ is not contained in $L_1$, since in that case as shown in Lemma 1, Axiom 2 is violated. Axiom 3 implies that $W$ is not a purely finitely additive measure either; as shown in Chichilnisky (1996b, 1997), the only possible form is as represented above.[8] □

**Example 2.** As an illustration of the representation theorem presented above, consider the case when the states are discrete, indexed by the integers $Z$ rather than $R$. For each real number $\mu$, $0 < \mu < 1$, a continuous linear functional $\Psi: l_\infty \to R$ can be defined as follows:

$$\Psi(x) = \mu \sum_{s=1}^{\infty} \lambda^{-s} u(x(s)) + (1-\mu) \lim_{s \in Z} u(x(s)), \tag{4}$$

where $\lim_{s \in Z} u(x(s))$ is the (Hahn–Banach) extension of the continuous linear ''limit'' operator to the space $l_\infty$ of all bounded real valued functions on $Z$. The interpretation of the two parts of the function $\Psi$ in Eq. (4) is simple. The first part is an integral operator with an integrable kernel $\{\lambda^{-s}\}_{s \in Z}$ that defines a countably additive measure on $Z$ and therefore emphasizes the weight of large-probability

[8] The connection between the function $W(x)$ and the Prospect theory of Kahneman and Tversky is the subject of another article.

events in the ranking of a lottery $x \in l_\infty$. The second part defines a purely finitely additive measure on $Z$, which assigns positive weight to ''small-probability'' events. It defines a measure with ''heavy tails''. Both parts are present, so $\Psi$ is sensitive to small and large-probability events. Catastrophic risks are therefore ranked more realistically by such functionals.

## 7. More examples and open questions

### 7.1. Examples

Consider an electrical utility such as Con Edison in New York. They seek to implement a production and service plan that would be optimal under normal conditions, while at the same time avoiding a potentially catastrophic ''black out'' incident that could be costly in monetary terms and in human lives. Following Axioms 1–3 and Theorem 1, a typical criterion that would be adopted would involve choosing among all possible plans so as to maximize the expected electrical throughput plus minimizing the probability of reaching a critical level beyond which there would be a ''black out''. It can be shown that such a criterion would satisfy Axioms 1–3.

### 7.2. The Allais paradox

The first and perhaps most famous violation of the expected utility models of choice under uncertainty is due to M. Allais, who showed experimental evidence that is inconsistent with NM axioms. A variation of this paradox was reported by Kahneman and Tversky (Tversky and Wakker, 1995). They defined four lotteries A, B, C and D, and observed that 82% of the subjects chose gamble A over another gamble B, and 83% of the subjects chose a gamble C over another D, so that at least 65% chose B and C. However, as shown below, the pair of gambles B and C is inconsistent with NM model of expected utility:

**Example 3.** Gamble A consists of a 0.33 chance of winning US\$2500, 0.66 chance of winning US\$2400, and 0.01 chance of winning US\$0, while gamble B has one chance of winning US\$2400. Gamble C consists of 0.33 chance of winning US\$2500, and 0.67 chance of winning US\$0, while gamble D consists of 0.34 chance of winning US\$2400 and 0.66 chance of winning US\$0.

Observe that if an individual prefers B over A, under the expected utility assumption this means that their (sure) utility function $u$ over income satisfies

$$u(2400) > 0.33u(2500) + 0.66u(2400) + 0.01u(0)$$

or

$$0.34u(2400) > 0.33u(2500) + 0.01u(0), \tag{5}$$

the latter of which contradicts a choice in favor of C because choosing C over D implies

$$0.33u(2500) + 0.67u(0) > 0.34u(2400) + 0.66u(0). \tag{6}$$

This contradiction is what is called an ''Allais paradox''.

One way to resolve this paradox is to understand that, when our new axioms are taken into consideration, the individual's utility function $u$ has two components in cases of small-probability events: one of these components is expected utility (as above), and the other is focused on the small-probability event $u(0)$. Therefore, inequality (5) above can now be written as:

$$0.34u(2400) > 0.33u(2500) + 0.01u(0) - \theta,$$

for some real number $\theta > 0$, representing a higher weight given to the low-probability (0.01) event of winning US\$0 than would be the case with expected utility. This implies:

$$0.33u(2500) + 0.67u(0) - \theta < 0.34u(2400) + 0.66u(0),$$

which is no longer inconsistent with Eq. (6). With the new axioms, therefore, Eq. (5) no longer contradicts Eq. (6) and the Allais paradox has been ''resolved''.

### 7.3. Open questions

· Risk aversion is typically defined with respect to the utility function that appears inside the expected utility functional (1). Here, this definition may not work, and an alternative definition may be needed. An interesting open question is on how to define risk aversion for the functionals in Eq. (1), which satisfy our axioms.

· Another question is how to define repeated game solutions (e.g. Nash equilibrium) that involve players with welfare functions of the forms identified here and to explore when these solutions exist.

· The traditional calculus of variation is based on integral operators that have ''finite'' kernels, such as exponential weight functions of the form $e^{-\nu s}$. This specification no longer holds here, and therefore the optimization of the operators emerging from the new axioms requires a new form of calculus of variation. It is of interest that standard tools of calculus of variations must be redeveloped in new directions. Some results already exist, e.g. Chichilnisky (1996b, 1997), but much work is still needed. The study of optimal solutions of these type of functionals has led to asymptotically autonomous dynamical systems that occur naturally when one extends the Euler Lagrange analysis of optimal solutions to encompass the type of operators defined here. Statistical analysis of such systems also requires new tools.

## Acknowledgements

I thank Tony Fisher, Geoffrey Heal and an anonymous referee for valuable comments and suggestions.

## References

Allais, M., 1988. The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function. In: Munier, B.R. (Ed.), Risk, Decision, and Rationality. Reidel, Dordrecht The Netherlands, pp. 233–289.

Arrow, K., 1971. Essays in the Theory of Risk Bearing. North-Holland, Amsterdam.

Cass, D., Chichilnisky, G., Wu, H., 1996. Individual risks and mutual insurance. Econometrica 64 (2), 333–341.

Chichilnisky, G., 1996a. Updating Von Neumann Morgenstern axioms for choice under catastrophic risks. In: Invited Presentation, The Fields Institute for Mathematical Sciences, June 9–11, 1996, Workshop on Catastrophic Environmental Risks, University of Toronto, Canada. .

Chichilnisky, G., 1996b. An axiomatic approach to sustainable development. Social Choice and Welfare 13, 257–321.

Chichilnisky, G., 1997. What is sustainable development. Land Economics 73 (4), 467–491, November 1997.

Hernstein, N., Milnor, J., 1953. An axiomatic approach to measurable utility. Econometrica 21, 291–297.

Machina, M., 1982. Expected utility analysis without the independent axiom. Econometrica 50, 277–323.

Machina, M., 1989. Dynamic consistency and non-expected utility models of choice under uncertainty. Journal of Economic Literature 22, 1622–1668, December.

Tversky, A., Wakker, P., 1995. Risk attitudes and decision weights. Econometrica 63 (6), 1225–1280, November.