Markets, Information, and Uncertainty

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CHAPTER 5

Existence and optimality of a general equilibrium with endogenous uncertainty

Graciela Chichilnisky

1 Introduction

Kenneth Arrow once said that uncertainty about prices may be the most important form of economic uncertainty. Yet the treatment of uncertainty in Arrow–Debreu markets reflects only nature's moves. It therefore neglects price uncertainty, because prices depend on human behavior.

This chapter attempts to close the gap. It defines a new concept of general equilibrium in markets where traders are uncertain about prices, and proves the existence of such an equilibrium. Traders do not know the possible equilibrium prices a priori. The state space which represents price uncertainty, and the financial instruments used to hedge this uncertainty, are all defined endogenously as part of a market equilibrium.

To motivate the problem, I show in Proposition 1 that trying to hedge price uncertainty within an Arrow–Debreu framework leads to paradoxical outcomes, which are connected with Russell's paradox in logics. Thus a new framework is needed.

The framework introduced here is similar to that of Arrow and Debreu in that there are several markets, several traders who act competitively, and all contracts are entered simultaneously. However, the treatment of uncertainty is different. It is given by "layers" of uncertainty, where each layer is logically conditional on the previous one.[1] Each layer

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[1] This is similar to compound lotteries, which the Von Neumann axioms require should be equivalent to standard lotteries. However, the compound lotteries lead here to market structures different from standard lotteries because in the model introduced here

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is a formalization of index-based securities markets which are widely traded today. They provide a conceptual explanation of the role of derivative securities and of their market organization. The states in the first layer represent all market clearing prices for commodities, the states in the second layer all market clearing prices for index-based securities, the states in the third layer are market clearing prices in markets which trade contingent on the prices of the indexes, and so forth.

The resulting economy expands the theory of markets to allow the states and the financial structure to be endogenously defined at an equilibrium, as a result of market forces. Each "layer" of uncertainty requires a constraint that is similar to a margin requirement. This is a realistic feature, and one that makes the economy quite different from that of Arrow and Debreu.

Theorem 1 establishes the existence of an equilibrium consisting of a "tree" of states representing uncertainty, the corresponding asset markets, and market clearing prices. The equilibrium allocation clears all markets, is fully insured, and is Pareto efficient.

1.1 *Motivating endogenous uncertainty*

Imagine an Arrow–Debreu economy facing several states of nature, with a complete set of asset markets to hedge nature's moves. For simplicity the economy has finitely many equilibria.[2] In a departure from the standard framework, the households anticipate that there may be several possible market clearing prices among which a random selection will be made. They do not know what these prices could be.

In addition to the states of nature, traders are now concerned about a new form of uncertainty, price uncertainty. This can be formalized by new "states" describing the possible market clearing prices.[3] These new states are endogenous to the functioning of the economy, whereas the states used in the Arrow–Debreu theory describe variables which are exogenous, such as the weather. If new assets are introduced to complete the market, the new augmented economy may have price uncertainty, because there may be several market clearing prices for the new assets themselves. This problem may reiterate, leading to a sequence of economies with an increasing number of asset markets, and gradually increasing state spaces. A first question is whether within an Arrow–Debreu framework traders can fully hedge all price risks. Proposition 1

there are several budget constraints akin to margin requirements, one for each "layer" of uncertainty.

2 This is a generic property, Debreu (1970).

3 See also Chichilnisky, Dutta, and Heal (1991).

shows that the Arrow–Debreu framework does not provide a satisfactory solution to the problem of price uncertainty. The Arrow–Debreu economy cannot hedge against its own price risks. One needs a new formalization for markets with price risks.

1.2 Expectations about prices

It seems useful to consider how price risks change traders' expectations and alter market behavior. Recall that in a standard Walrasian approach an auctioneer announces a vector of prices, and individuals choose asset holdings and consumption levels to maximize utility at those prices. Trade only occurs when demand equals supply, and all markets clear. This corresponds to individuals having *single valued* expectations about prices and leads to Pareto efficient allocations.

The problem is altered substantially when traders anticipate – or an auctioneer announces – that one of several possible market clearing prices will be chosen at random. Expectations about prices are now *multivalued* rather than *single valued*. The individuals' optimization problems are altered: Rather than choosing asset holdings to maximize utility at the equilibrium prices announced by the auctioneer, they choose so as to maximize expected utility, where the expectation is over *a set of several possible market clearing prices*. The old prices can no longer clear the markets, because the uncertainty faced now is different. The new market clearing prices reflect more complex behavior: The expected utility being maximized includes expectations about prices as well as about states of nature. The optimization problem solved by the traders is different, and therefore so are the solutions. This tallies with Proposition 1 below.[4]

2 Definitions

A pure exchange Arrow–Debreu E economy has l commodities, H traders indicated by h, and S "Savage states of nature." Each Savage state is a description of the environment arising from acts of nature and independent of the actions of the agents, a slight abuse of notation.[5] Let R =

[4] Hahn (1991) and Chichilnisky, Hahn, and Heal (1992) argue that correct anticipation of the Walrasian equilibrium prices is inconsistent with the new equilibrium when there are several equilibria prices. This tallies with the results of Chichilnisky, Dutta, and Heal (1991) and of Chichilnisky, Heal, Streufert, and Swinkels (1992) which argue, inter alia, that the correct anticipation of several market prices is inconsistent with an equilibrium having a price within this set.

[5] The most general interpretation of Savage states could incorporate price risks.

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$\{r_1, \ldots, r_B\}$ denote an $S \times B$ matrix of returns on the economy's assets which pay contingent on the Savage states. There is a complete set of assets to hedge against the acts of nature so that rank $(R) = S$. The initial endowment for each household h is denoted w^h and the economy's endowment is $w = \Sigma_h w^h$. Trader h has a strictly quasiconcave, C^2 (twice continuously differentiable) monotonically increasing Von Neumann–Morgenstern utility function $V^h : R^l \to R$ with nonzero gradients, and satisfying standard boundary conditions which ensure that the aggregate excess demand vector of the economy increases beyond any bound when a price goes to zero. Let $p \in R^{l \times S}$ denote a price vector, $ED(p)$ denote the excess demand function of the economy, and define the set of equilibrium prices

$$E(w) = \{p : ED(p) = 0\}$$

Definition 1: *An economy has price uncertainty*[6] *when* $E(w)$ *has cardinality* $N > 1$, *and trader h maximizes expected utility*

$$W^h\left(\left(x^{hi}\right)_{i=1,\ldots,N}\right) = EV^h\left(\left(x^{hi}\right)_{i=1,\ldots,N}\right)$$

where $i = 1, \ldots, N$ *are possible equilibrium prices and the expectation depends on a probability distribution over the set of prices* $\{1, \ldots, N\}$ *which is the same for all traders.*[7]

Assumption 1: *The economy has a finite set of equilibria for any set of initial endowments.*

This is satisfied by many exchange economies. More precisely, a family of utility functions, of which a residual set gives finitely many equilibria for any endowment, is the family of C^∞ functions whose bordered Hessians are nonzero everywhere. See Debreu (1970) and extensions – references are in Chichilnisky, Dutta, and Heal (1991).

3 An Arrow–Debreu economy cannot hedge its price risks

Let E be an economy with price uncertainty. Can we obtain an optimal (Pareto efficient) allocation of risk bearing by adding as many assets as needed to hedge against price uncertainty? Within a sequence economy the answer was provided in Hahn (1991), and in an Arrow–Debreu context it was provided by Chichilnisky, Hahn, and Heal (1992), Lemma 1. In both cases the answer is negative. In the following I briefly recall

[6] In the following the terms *price uncertainty* and *price risks* are used interchangeably.
[7] This is not strictly necessary but simplifies notation.

their arguments. As in Section 2, the economy E faces N states of "price uncertainty" and S Savage states, making for a space $\Sigma = N \times S$ of states of both types. The initial economy has a complete set of Arrow–Debreu contingent markets for exogenous uncertainty, that is, one for each element of S, so it is equivalent to an economy without exogenous uncertainty. Traders are concerned only with price risks and attempt to maximize expected utility as specified in Section 2. However, as traders are aware of the price risks, and no instruments are available to hedge these risks, the economy is "incomplete," in the sense that it has risks for which no hedge exists. The question is whether the Arrow–Debreu framework can be used to hedge price risks optimally.

If so, then all that would be needed to hedge price risks fully would be to introduce Arrow–Debreu contingent markets, one for each of the N price risks; optimal hedging would then ensue. In our example, we would need to introduce N new contingent markets, or alternatively, as shown in Arrow (1953), N Arrow securities, since there are N "price" risks. The new economy obtained from augmenting the old one is called C. The procedure of adding Arrow securities, also called "completing the market," always leads to optimal allocation of risk bearing in the case of exogenous risks. The following result shows that it does not work with endogenous risks. In other words, the Arrow–Debreu framework does not work for hedging endogenous risks.

Proposition 1: *An Arrow–Debreu economy with price risks cannot achieve optimal allocation of risk bearing by the introduction of Arrow–Debreu contingent markets or Arrow securities. No matter how many contingent markets or securities are introduced the augmented economy C has no Pareto efficient allocations, and therefore no competitive equilibrium.*

Proof: First observe that at each of the states $s \in S$, for all $i \in N$, the total endowments of society are the same. By assumption all households attach the same probability to the event that one given equilibrium price will occur. Under these conditions, at a Pareto efficient allocation, each household must consume the same Savage state dependent allocation across all states in the set N, that is, $x^{hsi} = x^{hsj}$ for each household h and all Savage states s, for any two price states $i,j = i, \dots, N$; for a proof see Chichilnisky, Hahn, and Heal (1992). Since for each $s \in S$, each household consumption across all states in the set N is the same, it follows that for each state $s \in S$ the price vectors dependent on the set of states N are all the same. But this implies that all market clearing prices are equal, so that there is no price uncertainty in the model, a contradiction. Since the contradiction arises from assuming that all price uncertainty can be hedged by a complete set of Arrow–Debreu price contingent markets, the proof is complete. QED.

4 Layers of uncertainty and the Russell paradox

We saw that an Arrow–Debreu economy cannot hedge price risks fully. Any attempt to complete the market by adding contingent markets or securities allocations fails. There are no Pareto efficient allocations. The failure can be viewed as the inability of the Arrow–Debreu economy to hedge against the price risks it generates.

A practical example will illustrate this failure and suggest an alternative market structure to hedge price risks. Consider a market in which oranges are traded forward. Assume that there are three possible market clearing prices for oranges, with the same probability each, and that this is common knowledge. In practice, to hedge against such price uncertainty, options on orange prices are introduced. This is how markets hedge against price uncertainty in concrete cases.

How are the market clearing prices determined? In an Arrow–Debreu economy all the market clearing prices are simultaneously determined for all states of nature by the auctioneer. When attempting to extend this procedure to our economy with price uncertainty a problem arises. An auctioneer cannot simultaneously determine the market clearing prices for oranges and for options on oranges.[8] This is because once the auctioneer announces any forward prices for oranges, there is no hedging role for the options on oranges. If, for example, the forward price for oranges announced by the auctioneer is \$2, then nobody will buy a call for oranges at a strike price \$$x$ if $x > 2$, and nobody will sell such a call if $x < 2$ unless paid at least the difference \$$2 - x$. At a strike price of \$2 the value of this option will be exactly zero. In other words: options on commodities do not have any role in allocation of price risks if they are traded simultaneously with forward commodity markets. Simultaneous trading across all states of uncertainty is of the essence in an Arrow–Debreu economy, so oranges at time t and their options are traded at once in such markets. This is the reason why an Arrow–Debreu economy cannot fully hedge price risks.

In practice, commodities at a given date are never traded at the same

[8] That is, the price of oranges at time t cannot be determined simultaneously with the price of options on oranges maturing at time t.

date as their options: The forward market for oranges is typically trade at a date posterior to that at which the option market closes, so that th price of oranges is still unknown when the option is traded. In othe words, there is a natural "ordering" in the markets for assets to hedg price uncertainty which cannot be formulated within the Arrow–Debre treatment of uncertainty, where all markets are simultaneous.

The ordering reflects the fact that the markets for those assets whos values depend on the prices of other assets will not improve risk alloca tion if the values of those underlying assets are revealed simultaneously The uncertainty must be revealed in an orderly fashion for these market to work together. There are "veils" of uncertainty which must be resolved in the proper order, and the time structure of trading takes care of this order. In our example, first the auctioneer must determine the price for the options *contingent on all the possible prices for oranges tomorrow*, and compute the corresponding aggregate demand for options. Only when market clearing prices have been found is the price for the underlying asset, forward oranges, realized. This argument leads to a nested sequence of ordered assets, and to orderly resolution of uncertainty. This is what we call here layers of price uncertainty, a treatment of uncertainty fundamentally different from that in the Arrow–Debreu economy.[9]

How then are assets to hedge price uncertainty to be traded? Rather than being contingent on several simultaneous states as in the Arrow–Debreu model, the assets are now defined in terms of nested risks, or layers of uncertainty. Each layer consists of *a set of states* which represent uncertainty of the same type, for example, uncertainty about all possible equilibrium prices for securities of a given type. All states within a layer are grouped together, and the uncertainty about a given layer is resolved by the assets of the following layer. I develop this concept formally in the next section.

To situate the problem within standard grounds and fix ideas, it is useful to draw an analogy between the problem of hedging endogenous uncertainty and the structure of the well-known Russell paradox. The solution to Russell's paradox led to the development of set theory as it is known today, see Halmos (1970). The paradox arises, for example, when we inquire whether a set is an element of itself, and can be illustrated as follows. A town has a barber who shaves all those who do not shave them-

[9] All uncertainty in the Arrow–Debreu economy derives from acts of nature, a type of uncertainty for which simultaneous contingent markets suffice to attain Pareto efficient allocations.

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selves. The question is: Does the barber shave himself? There exists no answer to this question; yes leads to no, and no leads to yes.

The solution to the paradox is to structure the universe into appropriate layers or logical "classes." When this is done, the question of whether a set belongs to itself is shown to be ill defined, so that it cannot be answered. Some objects are points and others are sets: Only points can belong to sets, whereas sets can only belong to higher level objects, called classes. The question about the barber is ill posed because it refers to a set as belonging, or not, to itself. Our informal language allows us to pose ill-defined questions.

An analogy between this chapter's problem and the Russell paradox is as follows. Consider an Arrow–Debreu economy with price uncertainty as defined above, where traders have set valued expectations about the possible price equilibria. Introduce all markets needed to hedge all risks, thus obtaining a "complete" market E in the sense of Arrow–Debreu, one in which all commodities and all assets are simultaneously traded. Does E hedge all its price risks? If it did, then as seen in Proposition 1 above E has only one equilibrium price in the first place, contradicting the hypothesis that it has price risks. If it does not, then we may introduce a new market to hedge any remaining price risk, a market not already in E. This is also a contradiction because, as defined, E contains all needed markets for hedging its risks. In reality, there is no logical answer to the question of whether E hedges all its price risks. This is the same problem as with the Russell paradox.

When trying to hedge against price uncertainty within an Arrow–Debreu economy in which all markets are traded simultaneously, one is attempting to obtain from the markets of this economy a hedge against the price risks that these markets generate themselves. As we saw above there is no logical solution to this problem: Our economic language allows us to pose an ill-defined question. Developing further the analogy with the Russell paradox, a solution could be provided by structuring the problem in logical "classes" or layers. One must structure uncertainty into layers, each layer designed to resolve the uncertainty created by previous ones, without ever attempting to go outside the logical order and ask any one layer to hedge against its own price risks. The next section follows this course of action to its logical conclusion.

5 A new economy with endogenous uncertainty

This section formalizes an economy in which uncertainty takes the form of a compound lottery represented by a "tree," or layered sets of states.

This representation of uncertainty is novel, and it leads to several "margin" requirements, one for each layer.

Within this economy I prove the existence of a general equilibrium in which all markets clear, where individuals maximize expected utility within the corresponding budgets, and where at an equilibrium all individuals are fully insured against price risks (Theorem 1). The intention is to obtain, within one single economy with layers of uncertainty, a result similar to that which has been obtained recently for a sequence of different economies by Chichilnisky, Dutta, and Heal (1991). They construct a sequence of different economies by progressively adding more financial markets and modifying the endowments of the traders, and show that full price insurance is achieved at the end of finitely many steps.[10]

The results of Chichilnisky, Dutta, and Heal (1991) were obtained by building a sequence of economies, each an enlargement of the previous one. This section shows that it is possible to obtain similar results working within *one single economy where households face a set of possible prices for each state of each layer, and where each household solves a single optimization problem.* In other words, by changing the structure of the uncertainty, I obtain results similar to those of Chichilnisky, Dutta, and Heal within a single economy rather than in a sequence of economies.

A new economy L is defined now as follows. It has H households denoted $h = 1, \ldots, H$, and l commodities. There are S states of nature or "Savage states." Each household h has an initial endowment vector $w^h \in R^{l \times S}$ of commodities contingent on states of nature. For the Savage states we have a complete set of assets, as required in Section 2. As before, each trader h has a preference over commodities, $V^h : R \to R$. Commodities contingent on Savage states are indicated by vectors $x \in R^{l \times S}$; when it is clear from the context, I also refer to these vectors as commodities.

[10] To achieve this they start from a Walrasian economy with several equilibrium prices of this first economy. The corresponding Walrasian equilibrium allocations are used as the endowments of a second economy, the endowments consisting of price contingent goods traded in price contingent markets where agents may now hedge against the price uncertainty of the first economy. The second step is to inform agents that this second economy has in turn several price equilibria. Using the Walrasian allocation of the second economy as initial endowments of a third economy, the agents are then allowed to add new commodities, new endowment, and new financial instruments. The procedure continues until an economy is reached in which there is no price uncertainty, which means that an economy with a unique Walrasian equilibrium is achieved. Chichilnisky, Dutta, and Heal (1991) prove that, under regularity assumptions, such an economy can be reached in a finite number of steps. This result depends on the regularity assumptions made in Section 2.

The economy L is therefore defined by its l commodities, S Savage states of nature, H traders, and their endowments and utilities:

$$L = \left\{X = R_+^l, s = 1, \ldots, S, w^h \in R^{l \times S}, V^h : X \to R, h = 1, \ldots, H\right\}$$

In addition to the Savage states there are states of price uncertainty in L. However, the actual market structure of L, namely what types of assets will be traded and how, is determined endogenously as part of the market equilibrium solution. The following determines the universe of "structures of uncertainty" in which the equilibrium structure of L will be pound.

5.1 *The structure of uncertainty in L*

A *structure of uncertainty* for L is defined by a finite set Y consisting of elements $y = 1, \ldots, Y$, each denoting a "layer" of uncertainty. For each layer y there is a set of states J^y, with J^y elements associated with and defined by the corresponding market clearing price for the market of the previous layer.

> **Example 1:** *In layer 1 there are J^1 states representing possible Arrow–Debreu market clearing prices for the $l \times S$ commodities; in level 2 there are J^2 possible prices for type 2 price indexes which pay contingent on the prices of the $l \times S$ commodities. Thus in the first two layers there are a total of $J^1 \times J^2$ possible equilibrium price combinations. The structure of uncertainty comes ordered into layers, and each state is contingent on the realization of previous states, thus describing an uncertainty "tree."*

The random variables describing uncertainty in L are paths through the uncertainty tree. A realization of random variable is called a *resolution of price uncertainty*. It is a vector consisting of Y states, one state from each of the Y layers. It is intended to represent a realization of one market clearing price for each of the Y layers. A resolution of price uncertainty is therefore a *realized path* of states and is represented by a Y dimensional vector $(j^1, \ldots, j^Y)$, where $\forall y = 1, \ldots, Y, j^y \in J^y$. The probability of the j^y state occurring within the set of states in the y-th layer is π_{j^y}, with $\Sigma_{z=1}^{J^y} \pi_{j^z} = 1$. The set Φ of resolutions of price uncertainty has therefore cardinality (denoted also Φ)

$$\Phi = \Pi_{z=1}^{Y} J^z \tag{5.1}$$

and each realized path $(j^1, \ldots, j^Y) \in \Phi$ occurs with a probability $\pi_{j^1 \ldots j^Y} = \pi_{j^1} \times \pi_{j^2} \times \ldots \times \pi_{j^Y}$. Figure 1 illustrates a tree and a realized path

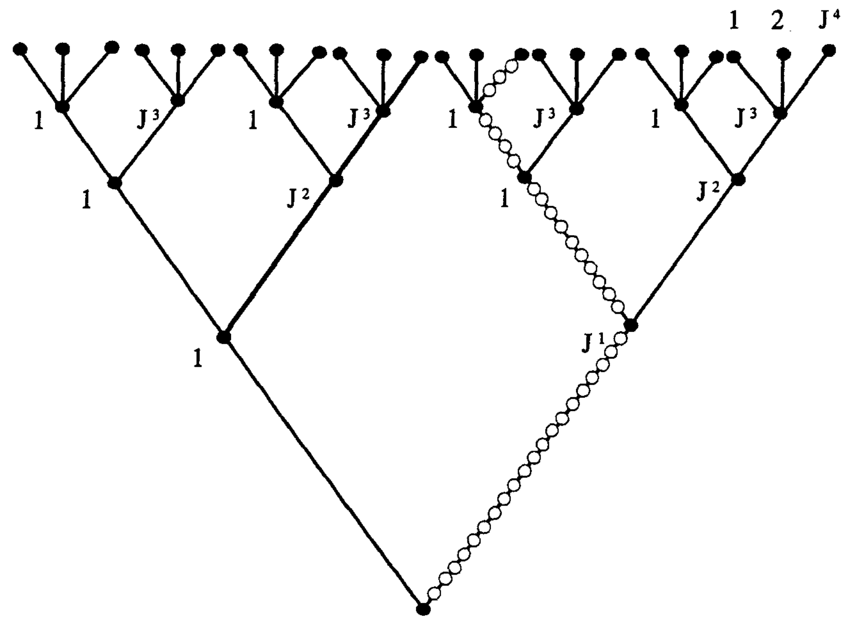


Figure 1. Four layers with two states in each of the first three layers and three states in layer 4 corresponding to the two equilibrium prices of the previous layer.

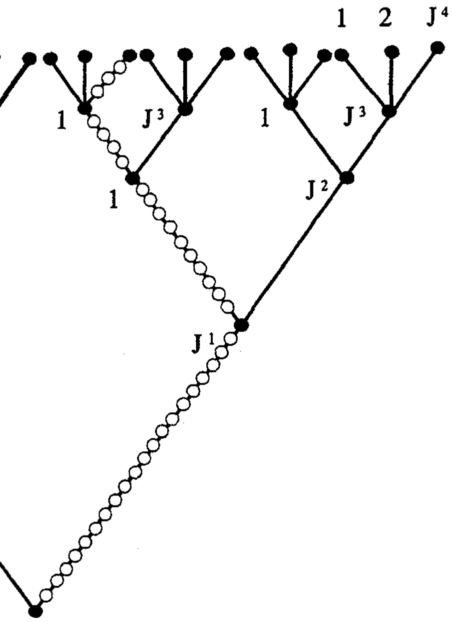
through the tree. There are four layers, and two possible equilibria in the markets defining each layer. At layer two, the four nodes of the tree correspond to the two equilibria in layer two markets, conditioned on each of the equilibria of the previous layer.

Summarizing all the above:

Definition 2: *A structure of uncertainty for the economy L is a list*

$$\left\{Y, J^y, y = 1, \dots, Y, \pi_{j^y} \text{ s.t.} \sum_{z=1}^{J^y} \pi_{j^z} = 1, j^y = 1, \dots, J^y\right\}$$

where the finite set Y represents the layers of uncertainty, each finite set J^y represents the states in layer y, and π_{j^y} is the probability of state j^y within the yth layer. A resolution of price uncertainty is a vector $(j^1, \dots, j^Y)$, where $j^y \in J^y$. The cardinality of the set of resolutions of price uncertainty is $\Phi = \Pi_{z=1}^{Y} J^z$.



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5.2 *The financial markets of L*

Turning now to the financial structure of the economy L, define an elementary y-asset as an instrument which allows the transfer of wealth among the states of the yth layer of uncertainty. Formally:

Definition 3: *An elementary y-asset is a vector $(1, \ldots, k, \ldots, 0) \in R^{J^y}$ which pays k units of numeraire in state $j^y \in J^y$ in exchange for 1 unit of the numeraire in state $j^1 \in J^y$, and 0 in all other states. A portfolio of y-assets is a linear combination of y-assets, a vector $\theta^{J^y} = (\theta_1, \ldots, \theta_{J^y}) \in R^{J^y}$ representing a transfer of wealth among the J^y states of the yth layer of uncertainty.*

Assumption 2: *For each layer $y = 1, \ldots, Y$ there exists a complete set of y-assets, that is, there are $J^y - 1$ distinct elementary y-assets for all $y = 1, \ldots, Y$.*

In the example illustrated in Figure 1 the resolution of the price uncertainty path $(j^1, \ldots, j^Y)$ is marked with bubbles. The uncertainty structure is described as follows. There are four layers of uncertainty, $Y = 4$. The number of states in each of the layers is[11]

$$J^1 = 2, J^2 = 2, J^3 = 2, J^4 = 3$$

The resolution of price uncertainty illustrated is the path

$$\left(j^1, \ldots, j^Y\right) = \left(2, 1, 1, 3\right)$$

with probabilities

$$\pi_{j^1 \ldots j^Y} = \left(1/2\right) \cdot \left(1/2\right) \cdot \left(1/3\right) = 1/24$$

Assumption 3: *Assume that each trader $h = 1, \ldots, H$ owns initially no assets in any state of price uncertainty, so that $\forall h$, h's portfolio of y-assets θ^{hj^y}, satisfies*

$$\sum_{i=1}^{J^y} \theta_i^h = 0 \; \forall y = 1, \ldots, Y \tag{5.2}$$

Definition 4: *A portfolio θ is an ex ante hedging strategy for the entire price uncertainty of the economy. It has Y layers, $\theta =$*

[11] To simplify the illustration we assumed that there are two equilibria in each of the first three layers, even though regular economies satisfying our assumptions will typically have an odd number of equilibria.

$([\theta^1], \ldots, [\theta^Y])$, each layer $[\theta^y]$ consisting of J^y different portfoli of $(y - 1)$-assets which hedge the price uncertainty of the pre\ ous layer, $y - 1$:

$$\theta = \left(\left[\theta^1\right], \ldots, \left[\theta^Y\right]\right) \text{ s.t. } \forall y = 1, \ldots, Y,$$
$$\left[\theta^y\right] = \left(\theta^{j^1}, \ldots, \theta^{J^y}\right),$$
$$\textit{with } \theta^{j^y} = \left(\theta^y_1, \ldots, \theta^y_{J^{y-1}}\right) \in R^{J^{y-1}} \; \forall j^y = 1, \ldots, J^y$$
$$\textit{and for each } y, \sum_{i=1}^{J^{y-1}} \theta^y_i = 0 \qquad (5.3$$

The hedging role of the portfolio θ can be explained intuitively a follows. For each $y = 1, \ldots, Y$ the yth layer of the portfolio $[\theta^y]$ consist: of one wealth transfer vector in $R^{J^{y-1}}$ for each of the J^y states in layer y indicating that there are J^y ways of insuring against the J^{y-1} states of pric uncertainty in layer $y - 1$, as defined above. Each J^{y-1}-dimensional vecto θ^{j^y} defines a $(j^y - 1)$-asset, that is, a transfer of wealth across the J^{y-1} states of layer $y - 1$ uncertainty. This indicates that the uncertainty introduced by the $(y - 1)$th layer is not hedged at this layer of uncertainty but rather at the next layer; furthermore, this uncertainty is hedged in J^y different ways, indicating that the hedging of the $(y - 1)$th layer of uncertainty has introduced in turn a new layer of uncertainty. This new layer y has J^y new states, each representing the possible market clearing prices of the $(y - 1)$th level markets.

A portfolio θ provides an ex ante investment plan for all possible resolutions of uncertainty $(j^1, \ldots, j^Y) \in \Phi$. Therefore, at each realized path of price uncertainty $(j^1, \ldots, j^Y) \in \Phi$, θ defines a portfolio path indicated

$$\theta\left(j^1, \ldots, j^Y\right) = \left(\theta^{j^1}_{j^0}, \ldots, \theta^{j^Y}_{j^{Y-1}}\right) \in R^Y \qquad (5.4)$$

where $\theta^{j^y}_{j^{y-1}} \in R$ is the realized value of the portfolio θ at the realized state j^y in layer y.

5.3 The trader

Turning now to the traders' behavior, a *plan* x^h for the h trader consists of an ex ante contract for each possible resolution of price uncertainty delivering an $l \times S$ vector at each state of each market layer. Therefore a trading plan is a vector $x \in R^{\Phi \times l \times S}$, where Φ is defined as in expression (5.1) above. For each resolution of price uncertainty $(j^1, \ldots, j^Y) \in \Phi$, the trading plan x^h of trader h defines *a path of Y net trade vectors in* $R^{l\times S}$, one vector in $R^{l\times S}$ for each state j^y in each layer y, denoted

$$x^h\left(j^1,\dots,j^Y\right) = \left(x^h_{j^1} - w^h, x^h_{j^2} - x^h_{j^1},\dots,x^h_{j^Y} - x^h_{j^{Y-1}}\right) \in R^{\Phi\times l\times S}$$

to indicate the net additions to the initial endowment of the trader w^h along the realized path $(j^1,\dots,j^Y)$. The trade at the 0 layer ($y = 0$) is by definition $x_{j^0} = w^h$.

5.4 *Prices*

Corresponding to trading plans $x \in R^{\Phi\times l\times S}$, an ex ante price system for the economy L is a vector $p \in R^{\Phi\times l\times S}$, listing the set of all market equilibrium prices at each layer of uncertainty. For each resolution of price uncertainty $(j^1,\dots,j^Y) \in \Phi$, p defines a *realized price path* $p(j^1,\dots,j^Y) = (p^{j^1},\dots,p^{j^Y}) \in R^{\Phi\times l\times S}$.

When price uncertainty is resolved a path $(j^1,\dots,j^Y)$ is realized and all the net trades in that path $x^h(j^1,\dots,j^Y)$ are realized. The total consumption vector of the household after each resolution of uncertainty is therefore the sum of the initial endowment w^h plus all the subsequent net trades in $x^h(j^1,\dots,j^Y)$, adding up to a total consumption vector $x^h_{j^Y}$

$$x^h_{j^Y} = w^h + \sum_{z=0}^{Y}\left(x^h_{j^z} - x^h_{j^{z-1}}\right) \in R^{l\times S}$$

where $(x^h_{j^z} - x^h_{j^{z-1}})$ is a net trade because $x^h_{j^{z-1}}$ is the endowment at layer z.

5.5 *Utilities*

Observe that the utility level trader h with plan x^h along the realized path $(j^1,\dots,j^Y)$ is the utility of the sum of all the net trade vectors along it plus the initial endowment

$$V^h\left(x^h\left(j^1,\dots,j^Y\right)\right) = V^h\left(w^h + \sum_{z=0}^{Y}\left(x^h_{j^z} - x^h_{j^{z-1}}\right)\right) = V^h\left(x^h_{j^Y}\right)$$

where the utility function V^h is as defined in Section 2. We may now define the utility functions of traders in the economy L over ex ante trading plans, which are the actions that traders take in this economy.

> **Definition 5:** *The utility derived by trader h from the ex ante trading plan* x^h *is the expected utility of consumption of* x^h *over all possible resolutions of uncertainty, namely over all paths* $(j^1,\dots,j^Y) \in \Phi$, *each path considered with its probability,* $\pi_{j^1\dots j^Y}$:

$$U^h\left(x^h\right) = EV^h\left(x^h\left(j^1, \ldots, j^Y\right)\right) \tag{5.5}$$

5.6 *Budgets and margins*

Definition 6: *For each price system p and portfolio θ^h, a budget set for the h trader is the set of all ex ante trading plans x^h which the trader can afford at all resolutions of price uncertainty:*

$$\begin{aligned} &B\left(p, \theta^h\right) = \Big\{x^h \text{ s.t. } \forall\left(j^1, \ldots, j^Y\right) \in \Phi, \\ &x^h\left(j^1, \ldots, j^Y\right) = \left(x^h_{j^1} - w^h, x^h_{j^2} - x^h_{j^1}, \ldots, x^h_{j^Y} - x^h_{j^{Y-1}}\right) \\ &\textit{satisfies } \textstyle\sum_{j^1=1}^{J^1} p^{j^1} . \left(x^h_{j^1} - w^h\right) = 0 \\ &\textit{and } p^{j^y} . \left(x^h_{j^y} - x^h_{j^{y-1}}\right) = \theta^{hj^{y+1}}_{j^y} \ \forall y = 1, \ldots, Y\Big\} \end{aligned} \tag{5.6}$$

This means for at any resolution of price uncertainty $(j^1, \ldots, j^Y)$, trader h may add a net trade vector $(x^h_{j^y} - x^h_{j^{y-1}}) \in R^{l \times S}$ to her/his endowment at the realized state j^y, provided its value computed at j^y prices p^{jy} does not exceed that of the trader's portfolio at that state, $\theta^{hy}_{j^{y-1}}$. This is a natural extension of the notion of a budget set in Arrow–Debreu theory, adapted to the structure of uncertainty in this model. It contains several constraints that are akin to "margin" requirements, as they limit the amount of trading on a given market as a function of the holdings on lower layers.

5.7 *An equilibrium of the economy* L

The next step is to define an equilibrium of the economy L. Recall that in addition to the usual variables describing an equilibrium, namely prices and trading levels, our equilibrium concept includes an *endogenous determination of the structure of uncertainty*. The structure of price uncertainty is defined by Y layers of uncertainty with J^y states in each layer, and the corresponding set of yth assets for all layers $y = 1, \ldots, Y$. Together with the structure of uncertainty, an equilibrium of L consists of a price vector p^* and, for each trader h, a trading plan x^{h*}, and a portfolio θ^{h*}, such that the consumption plan x^{h*} maximizes the utility $U^h(x^h)$ over all consumption plans within the budget set $B(p^*,\theta^{h*})$, given the plans of the other traders, $x^{h'}, \forall h' \neq h$; all markets clear, and all traders are fully insured against price risks.

Full insurance for price risks is formally defined as follows.

Definition 7: *The traders $h = 1, \ldots, H$ are fully insured against price risks at their consumption plans $\{x^h\}$, $h = 1, \ldots, H$, when $\forall h$, their total consumption, and therefore their utility levels $U^h(x^h)$ are the same at any realization of the layers of price uncertainty, that is, $\forall (j^1, \ldots, j^Y), (j^{1'}, \ldots, j^{Y'}) \in \Phi$*

$$x^{h*}_{j^Y} = w^h + \sum_{z=1}^{Y}\left(x^*_{j^z} - x^*_{j^{z-1}}\right) = x^{h*}_{j^{y'}}$$
$$= w^h + \sum_{z=1}^{Y}\left(x^*_{j^{z'}} - x^*_{j^{z-1'}}\right) \tag{5.7}$$

5.8 *Institutional structure: An illustration*

To fix ideas, I describe a possible institutional structure within which such an equilibrium may come about. This is to help the intuition and has no bearing on the formal definitions or the results. As in the Arrow–Debreu economy, one illustrates how an equilibrium emerges by imagining the actions of an auctioneer except that our auctioneer has a larger role than theirs.

The auctioneer announces here the structure of the price uncertainty in the second period, namely the number of layers of uncertainty Y, of states in each J^y, $y = 1, \ldots, Y$, and the probabilities π_{j^y} of each state j^y in J^y, with the corresponding financial markets.

For each such announcement, the auctioneer also provides an ex ante price system $p \in R^{\Phi \times l \times S}$ for the economy L. Using this information the traders, in turn, announce their portfolios θ^h and their ex ante plans $x^h \in R^{\Phi \times l \times S}$ within their budget sets $B(p, \theta^h)$. The auctioneer then reads the household plans; if an equilibrium obtains, trading is allowed. Otherwise the auctioneer tries again with another uncertainty structure, probabilities, and correspondingly new prices.

The auctioneer's role is to ensure that no trading takes place until all markets for commodities and for assets clear, and all households are fully insured against all price risks.

The existence of such an equilibrium seems like a tall order, but Theorem 1 below shows otherwise.

6 Existence of an equilibrium with full price insurance

Definition 8: *In the economy L defined above, the array $\{Y^*, J^{y^*}, x^{h*}, \theta^{h*}, p^*$ for $y = 1, \ldots, Y^*$ and $h = 1, \ldots, H\}$ is an equilibrium with full insurance against price uncertainty if for each trader h, the consumption plan x^{h*} maximizes the expected utility*

$$U^h\left(x^h\right) \tag{6.1}$$

over the budget set $B(p^*, \theta^{h*})$ given the consumption plans $x^{h'}$ of all other traders $\forall h' \neq h$, each trader h is fully insured against price risks, at each resolution of price uncertainty $(j_1, \ldots, j_Y) \in \Phi$ all asset markets to hedge price risks clear:

$$\sum_{h=1}^{H}\left(\theta^{h*}\right)_{j^{y-1}}^{j^y} = 0, \forall y = 1, \ldots, Y, \text{ where } \left(\theta^{h*}\right)_0^1 = 0 \tag{6.2}$$

and all commodity markets clear at each state of every layer of uncertainty:

$$\sum_{h=1}^{H}\left(x_{j^y}^{h*} - x_{j^{y-1}}^{h*}\right) = 0, \forall y = 1, \ldots, Y, \text{ where } x_{j^0}^{h*} = w^h \tag{6.3}$$

so that $\sum_{h=1}^{H} x_{j^Y}^{h*} - w^h = 0.$

Theorem 1: *The economy*

$$L = \left\{X = R_+^l, s = 1, \ldots, S, w^h \in R^{l\times S},\right.$$
$$\left.V^h : X \to R, h = 1, \ldots, H\right\}$$

as defined above has an equilibrium

$$\left\{Y^*, J^{y*} =, x^{h*}, \theta^{h*}, p^* \text{ for } y = 1, \ldots, Y^* \; h = 1, \ldots, H\right\}$$

with full insurance against price risks, and yielding a Pareto efficient allocation.

Proof: The proof proceeds by constructing the equilibria of a sequence of auxiliary economies, which are then discarded. There is no need to know the equilibria ex ante. Consider first an Arrow–Debreu economy $\{w^h, U^h : X \to R, h = 1, \ldots, H\}$ defined in Section 2, where the households are only concerned about the uncertainty defined by the Savage states $s = 1, \ldots, S$. Call this economy E_1. The set of Walrasian equilibria of E_1 is denoted $J^{1*} = \{1, \ldots, J^{1*}\}$; this set will define the first layer of price uncertainty of our economy $L, y = 1$. By definition, each of the J^{1*} equilibria of E_1 consists of a price vector $p^* \in R^{l\times S}$ and, for each h a consumption vector $x_{j^1}^{h*} \in R^{l\times S}$, for $j^1 = 1, \ldots, J^{1*}$.

Define now a second economy E_2 having the same H households, l commodities, and S Savage states as E_1. Assign E_2 a different commodity space and, for each h, different endowments and different utilities. The commodity space of E_2 has J^{1*} new states of uncertainty and therefore the commodity space is $R^{l \times S \times J^{1*}}$. In E_2 household h's endowment is the vector defined by the J^{1*} equilibria of E_1 side by side, that is, by the vector $(x_1^{h*}, \ldots, x_{J^1}^{h*}) \in R^{l \times S \times J^{1*}}$, where $x_{j^1}^{h*} \in R^{l \times S}$. Trader h's utility of consumption in E_2 as in equation (5.1) is the expected utility of consumption over the J^{1*} states, $V^h : R^{l \times S \times J^{1*}} \to R$, all states evaluated with the same probability:

$$V^h\left(y_1, \ldots, y_{J^1}\right) = \sum_{i=1}^{J^{1*}} \left(1/J^{1*}\right) U^h\left(y_i\right)$$

Assume now that the second economy E_2 has J^{2*} Walrasian equilibria. Then each of the J^{2*} Walrasian equilibria of E_2 consists of a price vector $p_{j^2}^* \in R^{l \times S \times J^{1*}}$ and, for each h, a consumption vector $x_{j^2}^{h*} \in R^{l \times S \times J^{1*}}$ for $j^2 = 1, \ldots, J^{2*}$. The set $J^{2*} = \{1, \ldots, J^{2*}\}$ of Walrasian equilibria of the economy E_2 defines layer $y = 1$ of uncertainty of our economy L.

E_2 has new states of uncertainty over and above those of E_1, indeed J^{1*} of them, but it also has all instruments needed to hedge this uncertainty, because, by construction, in E_2 there are markets contingent on the J^{1*} states of price uncertainty. The financial instruments corresponding to these contingent trades correspond to the portfolios of *1*-assets defined above, namely vectors describing wealth transfers between the J^{1*} price uncertainty states of economy E_1, $(\theta_1, \ldots, \theta_{J^{1*}})$, with $\Sigma_{i=1}^{J^{1*}} \theta_i = 0$. Since all assets needed to hedge the J^{1*} states of price uncertainty are available in E_2, at an equilibrium each trader h will achieve state independent consumption over the J^{1*} states. This is because in each of these J^{1*} states the total endowment $w = \Sigma_h w^h$ of the economy E_2 is the same, and every trader h has the same probability over the J^{1*} states.[12] Since each trader achieves state independent consumption over the J^{1*} states of price uncertainty, this means that at an equilibrium of E_2 the consumption vector $x_{j^2}^{h*} \in R^{l \times S \times J^{1*}}$ of the h trader consists of $S \times l$ coordinates repeated J^{1*} times. Clearly, this vector is then properly identified by $S \times l$ coordinates only, that is, $x_{j^2}^{h*} \in R^{l \times S}$. The corresponding prices are $p^{j^{2*}} \in R^{l \times S}$.

[12] This is the same point made in Proposition 1 above; the reader is referred to Chichilnisky, Dutta, and Heal (1991) for another proof.

Each trader in E_2 may shift wealth across the J^{1*} states to achieve the same consumption level at each, a shift represented by the vector with J^{1*} coordinates. At any market clearing equilibrium j^2 of E_2 this shift in wealth is, by definition, equal to a vector of differences between the value of the endowments evaluated at the equilibrium price $p^*_{j^2}$ in state j^2, namely $p^*_{j^2}.x^{h*}_{j^1}$, and the value of the equilibrium consumption at the same prices, namely, $p^*_{j^2}.x^{h*}_{j^2}$ for each $j^1 = 1, \dots, J^{1*}$. By definition of an equilibrium, each trader's consumption must be within his/her budget constraint, so that $\forall h = 1, \dots, H$,

$$p^{j^{2*}}.x^{h*}_{j^2} = \sum_{j^1=1}^{J^{1*}} \left(p^{2*}.x^{h*}_{j^1}\right) \text{ at each } j^2 = 1, \dots, J^{2*}$$

and that $\forall j^2 = 1, \dots, J^{2*}$

$$\sum_{j^1=1}^{J^{1*}} p^{2*}.\left(x^{h*}_{j^2} - x^{h*}_{j^1}\right) = 0 \tag{6.4}$$

Now define $[\theta^{h2}]$ as the following collection of J^{2*} vectors in $R^{J^{1*}}$:

$$\begin{aligned}\left[\theta^{h2}\right] &= \left(\theta_1^{h*j^2}, \dots, \theta_{J^1}^{h*j^2}\right)\\ &= \left(p^{2*}.\left(x^{h*}_{j^2} - x^{h*}_1\right), \dots, p^{2*}.\left(x^{h*}_{j^2} - x^{h*}_{J^1}\right)\right) \in R^{J^{1*}},\\ &\quad \text{for } J^2 = 1, \dots, J^{2*}\end{aligned} \tag{6.5}$$

Then by equation (6.4), $[\theta^{h2}]$ defines a layer 2 portfolio of 1-assets, since for each $j^2 = 1, \dots, J^{2*}$

$$\sum_{j^1=1}^{J^{1*}} \theta_i^{h*j^2} = 0$$

which is the condition required in the definition of a layer 2 portfolio, Section 5.

Recall that in the economy E_2 there are many different ways to achieve the equalization of consumption across the J^{1*} equilibria; there are precisely J^{2*} ways to do so, one for each of the equilibria of E_2. Corresponding to these are the J^{2*} portfolios of level 1 assets making the layer 2 portfolio $[\theta^{h2}]$ in equation (6.5). Since there are J^{2*} ways to achieve this equalization of consumption across all J^{1*} states of uncertainty, each yielding a different market clearing price or state in layer 2, E_2 introduces J^{2*} new states of price uncertainty which define our second layer

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coordinates. At any market clearing equi-
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hat $\forall h = 1,\dots,H$,

$$\left(p^{2*}.x^{h*}_{j^1}\right) \text{ at each } j^2 = 1,\dots,J^{2*}$$

$$J^{2*}$$

$$x^{h*}_{j^1}\Big) = 0 \tag{6.4}$$

following collection of J^{2*} vectors in $R^{J^{1*}}$:

$$,\dots,\theta^{h*j^2}_{J^1}\Big)$$
$$x^{h*}_{j^2} - x^{h*}_1\Big),\dots,p^{2*}.\left(x^{h*}_{j^2} - x^{h*}_{J^1}\right)\Big) \in R^{J^{1*}},$$
$$= 1,\dots,J^{2*} \tag{6.5}$$

(6.4), $[\theta^{h2}]$ defines a layer 2 portfolio of
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$y = 2$. To hedge these new states, consider a new economy E_3, which is defined exactly the same as E_2 but for its commodity space which is now equal to $R^{l\times S\times J^{2*}}$ to account for the fact that there are now J^{2*} new states of uncertainty. Repeating the same argument we build inductively a sequence of economies $\{E_y\}$, each economy E_y having the endowments provided by the set of J^{y-1*} equilibria of E_{y-1}, each economy E_y hedging the price risks of the former, E_{y-1}, and each trader h in E_y achieving state independent consumption over the states J^{y-1*}. This sequence of economies $\{E_y\}$ coincides with the sequence defined in Chichilnisky, Dutta, and Heal (1991).

To summarize: The economy E_y has a consumption set $R^{l\times S\times J^{y-1*}}$; trader h has as an initial endowment her/his allocation at the J^{y-1*} equilibria, namely the vector $(x^{h*}_1,\dots,x^{h*}_{J^{y-1*}}) \in R^{l\times S\times J^{y-1*}}$, where $x^{h*}_{j^{y-1*}}$ is the state-independent j^{y-1}th equilibrium allocation of trader h at the economy E_{y-1}. Trader h's utility of consumption in E_y is the expected utility of consumption over the J^{y-1*} states, $V^h: R^{l\times S\times J^{y-1*}} \to R$, all states evaluated with the same probability:

$$V^h\left(y_1,\dots,y_{J^{y-1}}\right) = \sum_{i=1}^{J^{y-1*}}\left(1/J^{y-1*}\right)U^h\left(y_i\right)$$

If the economy E_y has J^{y*} Walrasian equilibria, then each of the J^{y*} Walrasian equilibria of E_y consists of a price vector $p^*_{j^y} \in R^{l\times S\times J^{y-1*}}$ and, for each h, a consumption vector $x^{h*}_{j^y} \in R^{l\times S\times J^{y-1*}}$ for $j^y = 1,\dots,J^{y*}$. The set $J^{y*} = \{1,\dots,J^{y*}\}$ of Walrasian equilibria of the economy E_y defines the yth layer of uncertainty of our economy L. Since all assets needed to hedge the J^{y-1*} states of price uncertainty exist in E_y, households are fully insured against all the risk implicit in the J^{y-1*} states. This means that at an equilibrium of E_y the consumption vector $x^{h*}_{j^y} \in R^{l\times S\times J^{y-1*}}$ of the h trader consists of $S\times l$ coordinates repeated J^{y-1*} times. Clearly, this vector is then properly identified by $S\times l$ coordinates only, that is, $x^{h*}_{j^y} \in R^{l\times S}$ and the price $p^{j^{2*}} \in R^{l\times S}$.

Each trader in E_y shifts wealth across the J^{y-1*} states to achieve the same consumption level at each state, a shift represented by the vector with J^{y-1*} coordinates. At any market clearing equilibrium j^y of E_y this shift in wealth is, by definition, equal to a vector of differences between the value of the endowments evaluated at the equilibrium price $p^*_{j^y}$ in states j^y, namely $p^*_{j^y}.x^{h*}_{j^y}$. By definition of an equilibrium, each household's consumption must be within his/her budget constraint, so that $\forall h = 1, \dots, H$, and

$$p^{y*}.x^{h*}_{j^y} = \sum_{j^y=1}^{J^{y*}} \left(p^{y*}.x^{h*}_{j^{y-1}}\right) = 0 \text{ at each } j^y = 1, \dots, J^{y*} \tag{6.6}$$

Therefore $\forall j^y = 1, \dots, J^y*$

$$\sum_{j^{y-1}=1}^{J^{y-1}*} p^{jy*}.\left(x^{h*}_{j^y} - x^{h*}_{j^{y-1}}\right) = 0 \tag{6.7}$$

Now define $[\theta^{hy}]$ as the following J^y* vectors in $R^{J^{y-1}*}$:

$$\begin{aligned}\left[\theta^{hy}\right] &= \left(\theta_1^{h*j^y}, \dots, \theta_{J^1}^{h*j^y}\right) \\ &= \left(p^{jy*}.\left(x^{h*}_{j^y} - x^{h*}_1\right), \dots, \left(p^{jy*}.\left(x^{h*}_{j^y} - x^{h*}_{J^{y*}}\right)\right) \in R^{J^{y-1*}},\right. \\ &\quad \left. \text{for } j^y = 1, \dots, J^{y*}\right)\end{aligned} \tag{6.8}$$

Then by equation (6.7) $[\theta^{hy}]$ defines a layer y portfolio of $(y-1)$-assets, since for each $j^y = 1, \dots, J^y*$

$$\sum_{i=1}^{J^{y-1*}} \theta_i^{h*j^y} = 0 \tag{6.9}$$

which is the condition required in the definition of a layer y portfolio in Section 5.

Under the regularity assumption 2 of Chichilnisky, Dutta, and Heal (1991), which is also required here in Section 2, they proved that this process leads in a finite number of steps to an economy E_{Y*} having a unique, and Pareto efficient, Walrasian equilibrium.[13] In other words

[13] The result depends on the regularity of the economy, and the following fact: Any Pareto efficient allocation is the initial allocation of an Arrow–Debreu economy with a unique equilibrium, namely itself. Thus such economies have no price uncertainty. By regularity and the implicit function theorem, the number of equilibria is locally a continuous function of initial endowments. Therefore for all initial allocations in a neighborhood of a Pareto efficient allocation the economy has a unique equilibrium and thus no price uncertainty. The theorem in Chichilnisky, Dutta, and Heal (1991) shows that in a finite number of steps and by adding a finite number of assets, the initial endowment of the economy falls into the neighborhood of the Pareto frontier where the equilibrium is unique. Thus in a finite number of steps the process leads to an economy without price uncertainty. These, as shown in Proposition 1, are the only economies in which price uncertainty can be fully hedged within an Arrow–Debreu framework.

budget constraint, so that $\forall h = 1, \ldots, H$,

$\left(p^{y*}.x^{h*}_{j^{y-1}}\right) = 0$ at each $j^y = 1, \ldots, J^{y*}$

(6.6)

$, J^{y*}$

$- x^{h*}_{j^{y-1}}\big) = 0$ (6.7)

following J^{y*} vectors in $R^{J^{y-1*}}$:

$\big)$

$*\big), \ldots, \left(p^{jy*}.\left(x^{h*}_{j^y} - x^{h*}_{J^{y*}}\right)\right) \in R^{J^{y-1*}}$,

$ry*\big)$ (6.8)

6.7) $[\theta^{hy}]$ defines a layer y portfolio of

each $j^y = 1, \ldots, J^{y*}$

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$$\exists Y^* \text{ such that } x^{h*}_{j^Y} = x^{h*}_{j'^{Y*}}\ \forall j^{Y*}, j'^{Y*} = 1, \ldots, J^{Y*} \tag{6.10}$$

The existence of an equilibrium for the economy L can now be established. The uncertainty structure is defined by Y^* layers indexed by $y = 1, \ldots, Y^*$, with J^{Y*} states of uncertainty in each layer indicated $j^y = 1, \ldots, J^{y*}$. For each j^y define the probability $\pi_{j^y} = 1/J^{y*}$. For $y = 1, \ldots, Y^*$, consider $p^{j^{y*}} \in R^{l \times S}$ to be the j^y equilibrium price vector of the economy E_y, $j^y = 1, \ldots, J^{y*}$. Define

$$p^* = \left(p^{j^1*}, \ldots, p^{j^{Y*}}\right)_{j^1=1,\ldots,J^Y,\ldots,j^Y=1,\ldots,J^{Y*}} \in R^{\Phi \times l \times S} \tag{6.11}$$

Finally let $[\theta^{hy}]$ be defined as in equation (6.8), and define household h's ex ante portfolio θ^{h*} in the economy L to be:

$$x^{h*} = \left(\left(x^{h*}_{j^1} - w^h\right), \ldots, \left(x^{h*}_{j^{i+1}} - x^{h*}_{j^i}\right)\right)_{j^1=1,\ldots,J^Y,\ldots,j^Y=1,\ldots,J^{Y*}} \in R^{\Phi \times l \times S} \tag{6.12}$$

It remains now to check that $\{Y^*, J^{y*}, p^*, x^{h*}, \theta^{h*}, h = 1, \ldots, H, y = 1, \ldots, Y^*\}$ is an equilibrium of L.

First check that $\forall h = 1, \ldots, H, x^{h*}$ is in $B(p^*, \theta^{h*})$ as defined in equation (5.6). This follows from equations (6.4), (6.5), (6.7), (6.8), and (6.9).

Condition (6.3) for an equilibrium follows from the fact that for each $y = 1, \ldots, Y^*$ each market contingent on the J^{y-1} states of uncertainty of the economy E_{y-1} must clear at each Walrasian equilibrium j^y of the economy E_y; condition (6.2) follows directly from (6.5). Finally we check that U^h is maximized at x^{h*} given $x^{h'*}, \forall h' \neq h$. For this, recall that $x^{h*}_{j^Y} = x^{h*}_{j^{Y'}}, \forall j^Y, j^{Y'} = 1, \ldots, j^{Y*}$ by (6.5), so that traders are fully insured. Finally, note that the allocation $\{x^{h*}_{j^Y} \in R^{l \times S}, h = 1, \ldots, H\}$ is Pareto efficient because it is the Walrasian equilibrium of the economy E_{Y*}. This completes the proof. QED.

7 The literature on endogenous uncertainty

The problem of price uncertainty in general equilibrium was introduced and analyzed in two independent and simultaneous essays, each offering a different solution and both quite different from what is presented here: Hahn (1991) and Chichilnisky, Dutta, and Heal (1991). The results were elaborated further in Chichilnisky, Hahn, and Heal (1992). Hahn (1991) defines a two-period economy with incomplete markets for price risks. The agents alter their behavior when they learn about the several possible equilibrium prices, but have no more assets to hedge this uncertainty, so the market remains incomplete. Chichilnisky, Hahn, and Heal (1992) price the existence of an equilibrium with incomplete markets for price risks. In a different approach to the same problem, Chichilnisky, Dutta, and Heal (1991) construct a sequence of different, progressively larger economies in which new derivative securities are introduced at each stage, and show that this process leads in a finite number of steps to a new economy, the original augmented by markets for derivative securities, which has unique market clearing prices, and hence no price risks. Their analysis differs from that provided here in a number of ways. The first difference is that they consider a sequence of Arrow–Debreu economies, each having different endowments and utilities from the previous one, and at each step contracting takes place before the next economy is known. By contrast, in this chapter there is only one economy, and all contracting takes place simultaneously. The economy in this chapter has one utility function and one endowment vector for each trader. The agents in Chichilnisky, Dutta, and Heal (1991) anticipate correctly at each stage all the possible Walrasian equilibrium prices, an assumption I do not make in our definition of the economy with endogenous uncertainty in Section 5, or in the proof of existence of a market equilibrium, Theorem 2. Moreover, the concept of a market clearing equilibrium proposed here is different from that of an Arrow–Debreu economy in that I require "margins," or covered trading on the newly introduced markets. Finally, in contrast to Hahn (1991), Chichilnisky, Dutta, and Heal (1991), and Chichilnisky, Hahn, and Heal (1992), the optimal behavior of the agents with respect to the introduction of new states of price uncertainty is that agents choose their trading strategies so as to maximize utility, taking as given the behavior of others in the newly introduced markets.

An unusual feature of the type of uncertainty contemplated here is that it depends on the behavior of the agents as well as on acts of nature. In this sense the economy has endogenous uncertainty. Kurz (1993) discussed endogenous uncertainty in the context of a comment on Kesten–Stigum's model, and recently proposed a model where price expectations follow "rational beliefs," a special form of temporary equilibrium model. Expectations alter prices and therefore induce a well-known form of endogenous uncertainty, typical of "temporary

equilibrium" model. The concept of endog
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equilibrium" model. The concept of endogenous uncertainty was also discussed earlier in Hahn (1973) and in Dasgupta and Heal (1979). Within a three-period model, Henrotte (1992) has examined the role of options to hedge price uncertainty in securities markets. The first results on existence and characterization of markets with endogenous uncertainty in a general equilibrium framework were obtained in Chichilnisky and Wu (1991) and Chichilnisky, Dutta, and Heal (1991). Chichilnisky (1995) proves the existence of equilibrium in an economy where the state space varies with the production of the economy.

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CHAPTER 6

Market equilibrium wi
price uncertainty and

Peter H. Huang and Ho-Mo

1 Introduction

Arrow's (1953) classic two-period ge
model introduced the canonical theor
market behavior under uncertainty. The
by Arrow's formalization are those fact
such as hurricanes, earthquakes, droug
duction capabilities or consumer tastes. I
and Debreu (1959), with the introducti
gent commodity markets, reinterpret the
model of certainty in terms of a seque
reinterpretation allowed their results
optimality of static competitive equilibri
dynamic and uncertain world. Arrow's (
has become the standard role for secu
exogenous risks by shifting income acr
(1991) noted, Arrow (1953) and Arrow
financial economists with benchmarks fo
missing until then.

In Arrow's paradigm, uncertainty mea
possible states will prevail. Agents are as
able states that can arise. These states a
exclusive and exhaustive description of
ization of states of nature is related to, b

It is our pleasure to contribute this essay to hono
Arrow, from whose work, generosity, and scholarl
thank Ken Arrow, Don Brown, Graciela Chichilnis
Henrotte, Mordecai Kurz, Chris Shannon, Jan Wer
audiences of seminars at Columbia, Duke, and Sta