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CHAPTER 7

# Catastrophe futures: Financial markets for unknown risks

*Graciela Chichilnisky and Geoffrey Heal*

## 1 Introduction

New risks seem to be unavoidable in a period of rapid change. The last few decades have brought us the risks of global warming, nuclear meltdown, ozone depletion, failure of satellite launcher rockets, collision of supertankers, AIDS, and Ebola.[1] A key feature of a new risk, as opposed to an old and familiar one, is that one knows little about it. In particular, one knows little about the chances or the costs of its occurrence. This makes it hard to manage these risks. Existing paradigms for the rational management of risks require that we associate frequencies to various levels of losses. This poses particular challenges for the insurance industry, which is at the leading edge of risk management. Misestimation of new risks has led to several bankruptcies in the insurance and reinsurance businesses.[2] In this chapter we propose a novel framework for providing insurance cover against risks whose parameters are unknown. In fact many of the risks at issue may not be just unknown but also unknowable. It is difficult to imagine repetition of the events leading to global warming or ozone depletion, and therefore difficult to devise a relative frequency associated with repeated experiments.

A systematic and rational way of hedging unknown risks is proposed here, one which involves the use of securities markets as well as the more traditional insurance techniques. This model is consistent with the

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[1] A deadly viral disease.

[2] Many were associated with hurricane Andrew which at $18 billion in losses was the most expensive catastrophe ever recorded. Some of the problems which beset Lloyds of London arose from underestimating environmental risks.

current evolution of the insurance and reinsurance industries, which are beginning to explore the securitization of some aspects of insurance contracts via Act of God bonds, contingent drawing facilities, catastrophe futures, and similar innovations. Our model provides a formal framework within which such moves can be evaluated. An earlier version of this framework was presented in Chichilnisky and Heal (1993). Chichilnisky (1996) gives a more industry-oriented analysis.

This merging of insurance and the securities market is not surprising: Economists have traditionally recognized two ways of managing risks. One is risk pooling, or insurance, invoking the law of large numbers for independent and identically distributed (IID) events to ensure that the insurer's loss rate is proportional to the population loss rate. This will not work if the population loss rate is unknown. The second approach is the use of securities markets, and of negatively correlated events. This does not require knowledge of the population loss rate, and so can be applied to risks which are unknown or not independent. Securities markets alone could provide a mechanism for hedging unknown risks by the appropriate definition of states, but as we shall see this approach requires an unreasonable proliferation of markets. Using a mix of the two approaches can economize greatly on the number of markets needed and on the complexity of the institutional framework. In the process of showing this, we also show that under certain conditions the market equilibrium is anonymous in the sense that it depends only on the distribution of individuals across possible states, and not on who is in which state.

The reason for using two types of instruments is simple. Agents face two types of uncertainty: uncertainty about the overall incidence of a peril, that is, how many people overall will be affected by a disease, and then, given an overall distribution of the peril, uncertainty about whether they will be one of those affected. Securities contingent on the distribution of the peril hedge the former type of uncertainty; contingent insurance contracts hedge the latter.

Our analysis implies that insurance companies should issue insurance contracts which depend on the frequency of the peril, or statistical state. The insurance companies should offer individuals an array of insurance contracts, one valid in each possible statistical state. Insurance contracts are therefore contingent on statistical states. Within each statistical state probabilities are known, and companies are writing insurance only on known risks, something which is actuarially manageable. Individuals then buy the insurance they want between different statistical states via the markets for securities that are contingent on statistical states. The following is an illustration for purchasing insurance against AIDS, if the actuarial risks of the disease are unknown. One would buy insurance against AIDS by (1) purchasing a set of AIDS insurance contracts each of which pays off only for a specified incidence of AIDS in the population as a whole, and (2) making bets via statistical securities on the incidence of AIDS in the population. Similarly, one would obtain cover against an effect of climate change by (1) buying insurance policies specific to the risks faced at particular levels of climate change, and (2) making bets on the level of climate change, again using statistical securities. The opportunity to place such bets is provided in a limited way by catastrophe futures markets which pay an amount depending on the incidence of hurricane damage.

This chapter draws on recent findings of Chichilnisky and Wu (1991) and Cass, Chichilnisky, and Wu (1996), both of which study resource allocation with individual risks. Each of these essays develops further Malinvaud's (1972, 1973) original formulation of general equilibrium with individual risks, and Arrow's (1953) formulation of the role of securities in the optimal allocation of risk bearing. Our results are valid for large but finite economies with agents who face unknown risks and who have diverse opinions about these risks. In contrast, Malinvaud's results are asymptotic, valid for a limiting economy with an infinite population, and deal only with a known distribution of risks. The results presented here use the formulation of incomplete asset markets for individual risks used to study default in Chichilnisky and Wu (1991), Section 5.c. The risks considered here are unknown and possibly unknowable, and each individual has potentially a different opinion about these risks, whereas Chichilnisky and Wu (1991) and Cass, Chichilnisky, and Wu (1991) assume that all risk is known.

## 2 Notation and definitions

Denote the set of possible states for an individual by $S$, indexed by $s = 1, 2, \ldots, S$. Let there be $H$ individuals, indexed by $h = 1, 2, \ldots, H$. All households have the same state-dependent endowments. Endowments depend solely on the household's individual state $s$, and this dependence is the same for all households. The probability of any agent being in any state is unknown, and the distribution of states over the population as a whole is also unknown. A complete description of the state of the economy, called a *social state*, is a list of the states of each agent. A social state is denoted $\sigma$: it is an $H$ – vector. The set of possible social states is denoted $\Omega$ and has $S^H$ elements. A statistical description of the economy, called a *statistical state*, is a statement of the fraction of the population in each state. It is an $S$ – vector. As shown by Malinvaud (1973) there are $\binom{H+S-1}{S-1}$ statistical states. Clearly many social states map into a given statistical state. For example, if in one social state you are well and I am sick and in another, I am well and you are sick, then these two social states give rise to the same statistical state. Intuitively, we would not expect the equilibrium prices of the economy to differ in these two social states. One of our results shows that under certain conditions, the characteristics of the equilibrium are dependent only on the statistical state.

How does the distinction between social and statistical states contribute to risk management? Using the traditional approach, we could in principle trade securities contingent on each of the $S^H$ social states. This would require a large number of markets, a number which grows rapidly with the number of agents. However, the institutional requirements can be greatly simplified. When the characteristics of the equilibrium depend only on the statistical state, one can trade securities which are contingent on statistical states, that is, contingent on the distribution of individual states within the population, and still attain efficient allocations. This means that we trade securities contingent on whether 4% or 8% of the population are in state 5, but not on which people are in this state. Such securities, which we call *statistical securities*, plus mutual insurance contracts also contingent on the statistical state, lead (under the appropriate conditions) to an efficient allocation of risks. A mutual insurance contract contingent on a statistical state pays an individual a certain amount in a given individual state if and only if the economy as a whole is in a given statistical state.

Let $z_{jh\sigma}$ denote the quantity of good $j$ consumed by household $h$ in social state $\sigma$: $z_{h\sigma}$ is an $N$ dimensional vector of *all* goods consumed by $h$ in social state $\sigma$, $z_{h\sigma} = z_{jh\sigma}, j = 1, \ldots, N$, and $z_h$ is an $NS^H$ dimensional vector of *all goods consumed in all social states* by $h$, $z_h = z_{h\sigma}, \sigma \in \Omega$.[3]

Let $s(h, \sigma)$ be the state of individual $h$ in the social state $\sigma$, and $r_s(\sigma)$ the proportion of all households for whom $s(h, \sigma) = s$. Let $r(\sigma) = r_1(\sigma), \ldots, r_s(\sigma)$ be the distribution of households among individual states within the social state $\sigma$, that is, the proportion of all individuals in state $s$ for each $s$. $r(\sigma)$ is a statistical state. Let $R$ be the set of statistical states, that is, of vectors $r(\sigma)$ when $\sigma$ runs over $\Omega$. $R$ is contained in the $S$-dimensional simplex and has $\binom{H+S-1}{S-1}$ elements, see Malinvaud (1973) p. 385.

[3] Consumption vectors are assumed to be nonnegative.

$\Pi^h$ is household $h$'s probability distribution over the set of social states $\Omega$, and $\Pi^h_\sigma$ denotes the probability of state $\sigma$. Although we take social states as the primitive concept, we in fact work largely with statistical states. We therefore relate preferences, beliefs, and endowments to statistical states. This is done in the next section. Any distribution over social states implies a distribution over statistical states.

The following *anonymity assumption* is required:

$$r(\sigma) = r(\sigma') \rightarrow \Pi^h_\sigma = \Pi^h_{\sigma'}$$

This means that two overall distributions $\sigma$ and $\sigma'$ which have the same statistical characteristics are equally likely. Then $\Pi^h_\sigma$ defines a probability distribution $\Pi^h_r$ on the space of statistical states $R$. $\Pi^h_r$ can be interpreted, as remarked above, as $h$'s distribution over possible distributions of impacts in the population as a whole. The probability that a statistical state $r$ obtains and that simultaneously, for a given household $h$, a particular state $s$ also obtains, $\Pi^h_{sr}$, is[4]

$$\Pi^h_{sr} = \Pi^h_r r_s \text{ with } \sum_s \Pi^h_{sr} = \Pi^h_r \tag{2.1}$$

The probability $\Pi^h_s$ that, for a given $h$, a particular individual state $s$ obtains is therefore given by

$$\Pi^h_s = \sum_{r \in R} \Pi^h_r r_s$$

where $r_s$ is the proportion of people in individual state $s$ in statistical state $r$. Note that we denote by $\Pi^h_{s|r}$ the conditional probability of household $h$ being in individual state $s$, conditional on the economy being in statistical state $r$. Clearly $\Sigma_s \Pi^h_{s|r} = 1$. Anonymity implies that

$$\Pi^h_{s|r} = r_s$$

that is, that the probability of anyone being in individual state $s$ contingent on the economy being in statistical state $r$ is the relative frequency of state $s$ contingent on statistical state $r$.

## 3 The behavior of households

Let $e^h_s$ be the endowment of household $h$ when the individual state is $s$. We assume that household $h$ always has the same endowment in the indi-

[4] See Malinvaud (1973), p. 387, para. 1.

vidual state $s$, whatever the social state. We also assume that all households have the same endowment if they are in the same individual state. Endowments differ, therefore, only because of differences in individual states. This describes the risks faced by individuals.

Individuals have von Neumann–Morgenstern utilities:

$$W^h(z_h) = \sum_{\sigma} \Pi_{\sigma}^h U^h(z_{h\sigma})$$

This definition indicates that household $h$ has preferences on consumption which may be represented by a "state separable" utility function defined from elementary state-independent utility functions.

We assume like Malinvaud (1972) that *preferences are separable over statistical states*. This means that the utility of household $h$ depends on $\sigma$ only through the statistical state $r(\sigma)$. If we assume further that in state $\sigma$ household $h$ takes into account only its individual consumption, and what overall frequency distribution $r(\sigma)$ appears, and nothing else, then its consumption plan can be expressed as $z\sigma^h = z_{hsr}$: Its consumption depends only on its individual state $s$ and the statistical state $r$. Summation with respect to social states $\sigma$ in the expected utility function can now be made first within each statistical state. Hence we can express individuals' utility functions as:

$$W^h(z_{h\sigma}) = \sum_{r,s} \Pi_{sr}^h U^h(z_{hsr}) \tag{3.1}$$

which expresses the utility of a household in terms of its consumption at individual state $s$ within a statistical state $r$, summed over statistical states. This expression is important in the following results, because it allows us to represent the utility of consumption across social states $\sigma$ as a function of statistical states $r$ and individual states $s$ only. The functions $U_s^h$ are assumed to be $C^2$, strictly increasing, strictly quasiconcave, and the closure of the indifference surfaces $\{U_s^h\}^{-1}(x) \subset int\,(R^{N+})$ for all $x \in R^+$. The probabilities $\Pi_{\sigma}^h$ are in principle different over households.

## 4 Efficient allocations

Let $p^*$ be a competitive equilibrium[5] price vector of the Arrow–Debreu economy $E$ with markets contingent on all social states and let $z^*$ be the associated allocation. We will as usual say that $z^*$ is *Pareto efficient* if it is impossible to find an alternative feasible allocation which is preferred

[5] Defined formally below.

by at least one agent and to which no agent prefers $z^*$. Let $p\sigma^*$ and $z\sigma^*$ be the components of $p^*$ and $z^*$ respectively which refer to goods contingent on state $\sigma$.

We now define an Arrow–Debreu economy $E$, where markets exist contingent on an exhaustive description of all states in the economy, that is, for all social states $\sigma \in \Omega$. We therefore have $NS^H$ contingent markets. An *Arrow–Debreu equilibrium* is a price vector $p^* = (p_\sigma) \in R^{N\times\Omega}$, for each $\sigma$ $p_\sigma \in R^{N+}$, $\sigma \in \Omega$, and an allocation $z^*$ consisting of vectors $z_h^* = (z_{h\sigma}^*) \in R^{N\times\Omega}$, $z_{h\sigma}^* \in R^{N+}$, $\sigma \in \Omega$, $h = 1, \ldots, H$ such that for all $h$, $z_h^*$ maximizes

$$W^h\left(z_h^*\right) = \sum_\sigma \Pi_\sigma^h U^h\left(z_{h\sigma}^*\right) \tag{4.1}$$

subject to a budget constraint

$$p\left(z_h^* - e_h\right) = 0 \tag{4.2}$$

and all markets clear:

$$\sum_h\left(z_h^* - e_h\right) = 0 \tag{4.3}$$

Proposition 1 considers the case when households agree on the probability distribution over social states,[6] this common probability is denoted $\Pi$. It follows that they agree on the distribution over statistical states. In this case, the competitive equilibrium prices $p^*$ and allocations $z^*$ are the same across all social states $\sigma$, leading to the same statistical state $r$.[7]

> **Proposition 1:** *When agents have common probabilities, (see footnote 6) i.e., $\Pi^h = \Pi^j \forall h, j$, then equilibrium prices depend only on statistical states. Consider an Arrow–Debreu equilibrium of the economy E, $p^* = (p_\sigma^*)$, $z^* = (z_\sigma^*)$, $\sigma \in \Omega$. For every state $\sigma$ leading to a given statistical state r, that is, such that $r(\sigma) = r$, equilibrium prices and consumption allocations are the same, that is, there exists a price rector $p_r^*$ and an allocation $z_r^*$ such that $\forall\sigma$: $r(\sigma) = r$, $p_\sigma^* = p_r^*$, and $z_\sigma^* = z_r^*$, where $p_r^* \in R^{N+}$ and $z_r^* \in R^{NI}$ depend solely on r.*

[6] In a recent article, Klimper and Requate (1997) show that Proposition 1's proof holds also for households that do not agree on a common probability distribution over social states.

[7] Related propositions were established by Malinvaud in a simpler economy where all agents are identical, and risks are known.

*Proof:* In the Appendix. QED.

**Definition:** *An economy E is regular if at all equilibrium prices in E the Jacobian matrix of first partial derivatives of its excess demand function has full rank. Regularity is a generic property (Debreu (1970), Dierker (1982)).*

We now consider the general case, which allows for $\Pi^h \neq \Pi^j$ if $h \neq j$. Proposition 2 states that if the economy is regular, if all households have the same preferences and if there are two individual states, there is always one equilibrium at which prices are the same at all social states leading to the same statistical state. This confirms the intuition that the characteristics of an equilibrium should not be changed by a permutation of individuals: If I am changed to your state, and you to mine, everyone else remaining constant, then provided you and I have the same preferences, the equilibrium will not change.

**Proposition 2:** *Assume $\Pi^h \neq \Pi^k$ for some households h, k. When E is a regular economy, all agents have the same utilities,*[8] *and there are two individual states, then one of the equilibrium prices $p^*$ must satisfy $p^*_{\sigma 1} = p^*_{\sigma 2}$ for all $\sigma_1, \sigma_2$ with $r(\sigma_1) = r(\sigma_2)$.*

*Proof:* In the Appendix. QED.

## 5 Equilibrium in incomplete markets for unknown risks

Consider first the case where *there are no assets to hedge against risk*, so that the economy has incomplete asset markets. Individuals cannot transfer income to the unfavorable states. Examples are cases where individuals are not able to purchase hurricane insurance, as in some parts of the southeastern United States and in the Caribbean. Market allocations are typically inefficient in this case, since individuals cannot transfer income from one state to another to equalize welfare across states. Which households will be in each individual state is unknown. Each individual has a certain probability distribution over all possible social states $\sigma$, $\Pi^h$. In each social state $\sigma$ each individual is constrained in the value of her/his expenditures by her/his endowment [which depends on the individual state $s$ $(h,\sigma)$ in that social state]. In this context, a *general equilibrium of*

[8] The condition that all agents have the same preferences is not needed for this result. However, it simplifies that notation and the argument considerably. The general case is treated in the working papers from which this article derives.

*the economy with incomplete markets* $E_I$ consists of a price vector $p^*$ with $NS^H$ components and $H$ consumption plans $z_h^*$ with $NS^H$ components each, such that $z_h^*$ maximizes $W^h(z_h)$:

$$W^h(z_h) = \sum_\sigma \Pi_\sigma^h U^h(z_{h\sigma}) \tag{5.1}$$

subject to

$$p_\sigma(z_{h\sigma} - e_{h\sigma}) = 0 \text{ for each } \sigma \in \Omega \tag{5.2}$$

and

$$\sum_{h=1}^{H}(z_h - e_h) = 0 \tag{5.3}$$

The above economy $E_I$ is an extreme version of an economy with incomplete asset markets (see, for example, Geanakoplos 1990) because there are no markets to hedge against risks. There are $S^H$ budget constraints in equation (5.2).

## 6 Efficient allocations, mutual insurance, and securities

In this section we study the possibility of supporting Arrow–Debreu equilibria by combinations of statistical securities and insurance contracts, rather than by using state contingent contracts. As already observed, this leads to a very significant roduction in the number of markets needed. In an economy with no asset markets at all, such as $E_I$, the difficulty in supporting an Arrow–Debreu equilibrium arises because income cannot be transferred between states. On the basis of Propositions 1 and 2, we show that households can use securities defined on statistical states to transfer into each such state an amount of income equal to the expected difference between the value of Arrow–Debreu equilibrium consumption and the value of endowments in that state. The expectation here is over individual states conditional on being in a given statistical state. The difference between the actual consumption-income gap given a particular individual state and its expected value is then covered by insurance contracts. In the following, $A$ denotes the combinatorial number $A = \begin{pmatrix} H+S-1 \\ S-1 \end{pmatrix}$.

**Theorem 1:** *Assume that all households in E have the same probability* $\Pi$ *over the distribution of risks in the population.*

*Then any Arrow–Debreu equilibrium allocation* $(p^*, z^*)$ *of E (and therefore any Pareto optimum) can be achieved within the general equilibrium economy with incomplete markets* $E_I$ *by introducing a total of A mutual insurance contracts to hedge against individual risk, and A statistical securities to hedge against social risk. In a regular economy with two individual states and identical preferences, even if agents have different probabilities, there is always an Arrow–Debreu equilibrium* $(p^*, z^*)$ *in E which is achievable within the incomplete economy* $E_I$ *with the introduction of I.A mutual insurance contracts and A statistical securities.*

*Proof:* In the Appendix. QED.

### 6.1 *Market complexity*

We can now formalize a statement made before about the efficiency of the institutional structure proposed in Theorem 1 by comparison with the standard Arrow–Debreu structure of a complete set of state-contingent markets. We use complexity theory, and in particular the concept of *NP completeness*. The key consideration in this approach to studying problem complexity is how fast the number of operations required to solve a problem increases with the size of the problem.

> **Definition:** *If the number of operations required to solve a problem must increase exponentially for any possible way of solving the problem, then the problem is called "intractable" or more formally, NP-complete. If instead this number increases polynomially, the problem is "tractable."*[9]

The motivation for this defiuition is that if the number of operations needed to solve the problem increases exponentially with some measure of the size of the problem, there will be examples of the problem that no computer can or ever could solve. Hence there is no possibility of ever designing a general efficient algorithm for solving these problems. However, if the number of operations rises only polynomially then it is in principle possible to devise a general and efficient algorithm for the problem.

Theorem 2 investigates the complexity of the resource allocation problem in the Arrow–Debreu framework and compares this with the

[9] Further definitions are in Garey and Johnson (1979).

framework of Theorem 1. We focus on how the problem changes as the economy grows in the sense that the number of households increases, and consider a very simple aspect of the allocation problem, described as follows. Suppose that the excess demand of the economy $Z(p)$ is known. A particular price vector $p^*$ is proposed as a market clearing price. We wish to check whether or not it is a market clearing price. This involves computing each of the coordinates of $Z(p)$ and then comparing with zero. This involves a number of operations proportional to the number of components of $Z(p)$; we therefore take the rate at which the dimension of $Z(p)$ increases with the number of agents to be a measure of the complexity of the resource allocation problem. In summary: we ask how the difficulty of verifying market clearing increases as the number of households in the economy rises. We show that in the Arrow–Debreu framework this difficulty rises exponentially, whereas in the framework of Theorem 1 it rises only polynomially.

**Theorem 2:** *Verifying market clearing is an intractable problem in an Arrow–Debreu economy, that is, the number of operations required to check if a proposed price is market clearing increases exponentially with the number of households H. However, under the assumptions of Theorem 1, in the economy $E_I$ supplemented by I.A mutual insurance contracts and A statistical securities, verifying market clearing is a tractable problem, that is, the number of operations needed to check for market clearing increases only polynomially with the number of households.*

*Proof:* The number of operations required to check that a price is market clearing is proportional to the number of market clearing conditions. In $E$ we have $NS^H$ markets. Hence the number of operations needed to check if a proposed price is market clearing must rise exponentially with the number of households $H$. Consider now the case of $E_I$ supplemented by $I.A$ mutual insurance contracts and $A$ securities. Under the assumptions of Theorem 1, by Propositions 1 and 2, we need only check for market clearing in one social state associated with any statistical state, because if markets clear in one social state leading to a certain statistical state they will clear in all social states leading to the same statistical state. Hence we need to check a number of goods markets equal to $N.A$, plus markets for mutual insurance contracts and securities. Now

$$A = \binom{H + S + 1}{S - 1} = \Phi\left(H,S\right)$$

where $\Phi$ $(H,S)$ is a polynomial in $H$ of order $(S - 1)$. Hence $A$ itself is a polynomial in $H$ whose highest order term depends on $H^{S-1}$, completing the proof. QED.

## 7 Catastrophe futures and bundles

We mentioned in the introduction that securities contingent on statistical states are already traded as "catastrophe futures" on the Chicago Board of Trade, where they were introduced in 1994. (The concept was introduced and developed in Chichilnisky and Heal (1993).) Catastrophe futures are securities which pay an amount that depends on the value of an index (PCS) of insurance claims paid during a year. One such index measures the value of hurricane damage claims; others measure claims stemming from different types of natural disasters. The value of hurricane damage claims depends on the overall incidence of hurricane damage in the population, but is not affected by whether any particular individual is harmed. It therefore depends, in our terminology, on the statistical state, on the distribution of damage in the population, not on the social state. Catastrophe futures are thus financial instruments whose payoffs are conditional on the statistical state of the economy. They are statistical securities. According to our theory, a summary version of which appeared in Chichilnisky and Heal (1993), they are a crucial prerequisite to the efficient allocation of unknown risks. As the incidence and extent of natural disaster claims in the United States has increased greatly in recent years, risks such as property casualty due to hurricane risks are in effect unknown risks. Insurers are concerned that the incidence of storms may be related to trends in the composition of the atmosphere and incipient greenhouse warming. However, catastrophe futures are not on their own sufficient for this; they do not complete the market. Mutual or contingent insurance contracts, as described above, are also needed. These provide insurance conditional on the value of the catastrophe index. The two can be combined into "catastrophe bundles". See Chichilnisky (1996).

## 8 Conclusions

We have defined an economy with unknown individual risks and established that a combination of statistical securities and mutual or contingent insurance contracts can be used to obtain an efficient allocation of risk bearing. Furthermore, we have shown that this institutional structure is efficient in the sense that it requires exponentially fewer markets than the standard approach via state-contingent commodities. In fact, the state-contingent problem is "intractable" with individual risks (formally, NP-complete) in the language of computational complexity, whereas our approach gives a formulation that is polynomially complex. This greatly increases the economy's ability to achieve efficient allocations. Another interesting feature of this institutional structure is the interplay of insurance and securities markets involved. Its simplicity leads to successful hedging of unknown risks and predicts a convergence between the insurance and securities industries.

## 9 Appendix

**Proposition 1:** *When agents have common probabilities, that is, $\Pi^h = \Pi^j \forall h, j$, then equilibrium prices depend only on statistical states. Consider an Arrow–Debreu equilibrium of the economy E, $p^* = (p_\sigma{}^*)$, $z^* = (z_\sigma{}^*)$, $\sigma \in \Omega$. For every state $\sigma$ leading to a given statistical state r, that is, such that $r(\sigma) = r$, equilibrium prices and consumption allocations are the same, that is, there exists a price vector $p_r^*$ and an allocation $z_r^*$ such that $\forall\sigma$: $r(\sigma) = r$, $p_\sigma{}^* = p_r^*$ and $z_\sigma{}^* = z_r^*$ where $p_r^* \in R^{N+}$ and $z_r^* \in R^{NI}$ depend solely on r.*

*Proof:* Consider $\sigma_1$ and $\sigma_2$ with $r(\sigma_1) = r(\sigma_2) = r$. Note that the total endowments of the economy are the same in $\sigma_1$ and $\sigma_2$, both equal to $s_r = Hr_s e_{hs}$ (recall that $e_{hs} = e_s$ as endowments depend only on individual states and not on household identities). Also, by the anonymity assumption, $\Pi_{\sigma_1} = \Pi_{\sigma_2} = \Pi_r$, where $\Pi_r$ is the common probability of any social state in the statistical state $r$. Let $\Pi_{\sigma|r}$ be the probability of being in social state $\sigma$ given statistical state $r$. By the anonymity assumption on probabilities this is just $1/\#\Omega_r$. We now show that for every household $h$, $z^*_{h\sigma_1} = z^*_{h\sigma_2}$, due to the Pareto efficiency of Arrow–Debreu equilibria. Let $\Omega_r = \{\sigma : r(\sigma) = \sigma\}$. Let $z^* = (z^*_{h\sigma})$, and assume in contradiction to the proposition that there are $\sigma_1$ and $\sigma_2 \in \Omega_r$ such that $z^*_{h\sigma_1} \neq z^*_{h\sigma_2}$ for some $h$. Define $Ez_{hr} = \Sigma_{\sigma\in\Omega_r} z^*_{h\sigma}\Pi_{\sigma|r} = (1/\#\Omega_r)\,\Sigma_{\sigma\in\Omega_r z^*_{h\sigma}}$. This is the expected value of $(z^*_{h\sigma})$ given that the economy is in the statistical state $r$. Now

$$\sum_h Ez_{hr} = \sum_h \frac{1}{\#\Omega_r} \sum_{\sigma\in\Omega_r} z^*_{h\sigma} = \sum_h z^*_{h\sigma}$$

so that $Ez_{h\sigma}$ is a feasible consumption vector for each $h$ in the statistical state $r$. Next we show that by strict concavity, moving for each $h$ and each $\sigma$ from $z^*_{h\sigma}$ (which depends on $\sigma$) to $Ez_{hr}$ (which is the same for all $\sigma \in \Omega$) is a strict Pareto improvement. This is because

$$W^h\left(z^*_{h\sigma}\right) = \sum_{\sigma}\Pi_\sigma U^h\left(z^*_{h\sigma}\right) = \sum_r \Pi_r \sum_{\sigma\in\Omega}\Pi_{\sigma|r}U^h\left(z^*_{h\sigma}\right)$$

By strict concavity of preferences,

$$\sum_r \Pi_r \sum_{\sigma\in\Omega_r}\Pi_{\sigma|r}U^h\left(z^*_{h\sigma}\right) <$$

$$\sum_r \Pi_r \sum_{\sigma\in\Omega_r} U^h\left(\sum_{\sigma\in\Omega} z^*_{h\sigma}\Pi_{\sigma|r}\right) = \sum_r \Pi_r \sum_{\sigma\in\Omega} U^h\left(Ez_{h\sigma}\right)$$

Since $Ez_{h\sigma}$ is Pareto superior to $z^*$ with $z^*_{h\sigma_1} \neq z^*_{h\sigma_2}$, such a $z^*$ cannot be an equilibrium allocation. Hence $z^*_{h\sigma_1} = z^*_{h\sigma_2} = z^*_{hr}$ for all $h = 1, \ldots, H$. Note that this implies that in an equilibrium, household $h$ consumes the same allocation $z^*_{hr}$ across all individual states $s$, that is, it achieves full insurance. Since $p^*$ supports the equilibrium allocation $z^*$, and $z^*_{h\sigma_1} = z^*_{h\sigma_2}$ it follows that $p^*_{\sigma_1} = p^*_{\sigma_2}$ when $r(\sigma_1) = r(\sigma_2)$, because utilities are assumed to be $C^2$ and, in particular, to have a unique gradient at each point which, by optimality, must be collinear both with $p^*_{\sigma_1}$ and with $p^*_{\sigma_2}$, that is, $p^*_{\sigma_1} = p^*_{\sigma_2} = p^*_r$. This implies that at an equilibrium, household $h$ faces the same prices $p^*_r$ at any $\sigma$ with $r(\sigma) = r$. QED.

**Proposition 2:** *Assume $\Pi^h \neq \Pi^k$ for some households $h$, $k$. When $E$ is a regular economy, all agents have the same utilities,*[10] *and there are two individual states, one of the equilibrium prices $p^*$ must satisfy $p^*_{\sigma_1} = p^*_{\sigma_2}$ for all $\sigma_1,\sigma_2$ with $r(\sigma_1) = r(\sigma_2)$.*

*Proof:*

Assume that $E$ is regular, that all agents have the same preferences, and that $S = 2$. Consider two social states $\sigma_1$ and $\sigma_2$ with $r(\sigma_1) = r(\sigma_2)$, and such that $\sigma_1$ differs from $\sigma_2$ only on the individual states of the two households $h_1$ and $h_2$ which are permuted, that is, $s(h_1,\sigma_1) = s(h_2,\sigma_2)$ and $s(h_2,\sigma_1) = s(h_1,\sigma_2)$. Assume

[10] The condition that all agents have the same preferences is not needed for this result, but simplifies the notation and the proof considerably. In the working papers from which this article derives, the general case was covered. See also footnote 6 for the case where agents have different probabilities.

that there exists an equilibrium price for $E, p^* \in R^{NS^H}$, such that its components in states $\sigma_1$ and $\sigma_2$ are different, that is, $p^*_{\sigma_1} \neq p^*_{\sigma_2}$. Define now a new price $p^*_c \in R^{NS^H}$, called a "conjugate" of $p^*$, which differs from $p^*$ only in its coordinates in states $\sigma_1$ and $\sigma_2$, which are permuted as follows: $\forall\ \sigma \neq \sigma_1, \sigma_2$, $\overline{p}\sigma^* = p^*_\sigma$, $\overline{p}^*_{\sigma_1} = p^*_{\sigma_2}$, and $\overline{p}^*_{\sigma_2} = p^*_{\sigma_1}$. We now show that $\overline{p}^*$ is also an equilibrium price for the economy $E$. At $\overline{p}^*$, household $h_1$ has the same endowments and faces the same prices in states $\sigma_1$ and $\sigma_2$ as it did at states $\sigma_2$ and $\sigma_1$ respectively at price $p^*$; at all other states $\sigma \in \Omega$, $h_1$ faces the same prices and has the same endowments facing $p^*$ and facing $\overline{p}^*$. The same is true of household $h_2$. Furthermore $h_1$ and $h_2$ have the same utilities and probabilities at $\sigma_1$ and $\sigma_2$ because $r(\sigma_1) = r(\sigma_2)$ and probabilities are anonymous. Therefore the excess demand vectors of $h_1$ in states $\sigma_1$ and $\sigma_2$ at prices $p^*$ equal the excess demand vectors of $h_2$ in $\sigma_2$ and $\sigma_1$ respectively at prices $\overline{p}^*$, and at all other states $\sigma \in \Omega$ the excess demand vectors of $h_1$ are the same at prices $p^*$ and $\overline{p}^*$. Reciprocally, the excess demand vectors of $h_2$ in $\sigma_1$ and $\sigma_2$ at prices $p^*$ equal the excess demand vectors of $h_1$ in $\sigma_2$ and $\sigma_1$ respectively at prices $\overline{p}^*$, and in all other states $\sigma$, the excess demand vectors of $h_2$ are the same as they are with prices $p^*$. Formally:

$$z_{h_1\sigma_1}(\overline{p}^*) = z_{h_2\sigma_2}(p^*),\ z_{h_1\sigma_2}(\overline{p}^*) = z_{h_2\sigma_1}(p^*)$$
$$z_{h_2\sigma_1}(\overline{p}^*) = z_{h_1\sigma_2}(p^*),\ z_{h_2\sigma_2}(\overline{p}^*) = z_{h_1\sigma_1}(p^*)$$

and $\forall \sigma \in \Omega, \sigma \neq \sigma_1, \sigma_2$:

$$z_{h_1\sigma}(p^*) = z_{h_1\sigma}(\overline{p}^*),\ z_{h_2\sigma}(p^*) = z_{h_2\sigma}(\overline{p}^*)$$

The excess demand vectors of all other households $h \neq h_1, h_2$ are the same for $p^*$ and $\overline{p}^*$. Therefore at $\overline{p}^*$ the aggregate excess demand vector of the economy is zero, so that $\overline{p}^*$ is an equilibrium. The same argument shows that permuting the two components $p^*_{\sigma_1}, p^*_{\sigma_2}$ of a price $p^*$ at any two social states $\sigma_1, \sigma_2$ leading to the same statistical state $r(\sigma_1)$ leads from an equilibrium price $p^*$ to another equilibrium price $\overline{p}^*$. This is because if two social states $\sigma_1$ and $\sigma_2$ lead to the same statistical state and there are two individual states $s_1$ and $s_2$ then there is a number $k > 0$ such that $k$ households who are in $s_1$ in $\sigma_1$ are in $s_2$ in $\sigma_2$ and another $k$ households who were in $s_1$ in $\sigma_2$ are in $s_2$ in $\sigma_1$, while remaining in the same individual states otherwise. These two sets of $k$ households can be paired. For every pair of households, the above argument applies. Hence it applies to the sum of the demands, so that the new price $\bar{p}^*$ is an equilibrium.

Now consider any regular economy $E$ with a finite number of equilibrium prices denoted $p_1^*, \ldots, p_k^*$. We shall show that there exists a $j \leq k$ s.t. $p_j^*$ assigns the same price vector to all social states $\sigma_1,\sigma_2$ with $r(\sigma_1) = r(\sigma_2)$. Start with $p_1^*$; if $p_1^*$ does not have this property, consider the first two social states $\sigma_1,\sigma_2$ with $r(\sigma_1) = r(\sigma_2)$ and $p_{1\sigma_1}^* \neq p_{1\sigma_2}^*$. Define $\bar{p}_1^*$ as the conjugate of $p_1^*$ constructed by permuting the prices of the social states $\sigma_1$ and $\sigma_2$. If $\forall j > 1, p_j^* = \bar{p}_1^*$, then there are two price equilibria, that is, $k = 2$; since, however, the number of price equilibria must be odd,[11] there must exist $p_{j_1}^*$ with $j_1 > 1$ and $p_{j_1}^* \neq \bar{p}_1^*$. Consider now the conjugate of $p_{j_1}^*$ with respect to the first two social states $\sigma_1,\sigma_2$ which correspond to the same statistical state and have different components in $p_{j_1}^*$, and denote this conjugate $\bar{p}_{j_1}^*$. Repeat the procedure until all equilibria are exhausted. In each step of this procedure, two different price equilibria are found. Since the number of equilibria must be odd, it follows that there must exist a $j \leq k$ for which all conjugates of $p_j^*$ equal $p_j^*$. This is the required equilibrium which assigns the same equilibrium prices $p_{\sigma_1}^* = p_{\sigma_2}^*$ to all $\sigma_1,\sigma_2$ with $r(\sigma_1) = r(\sigma_2)$, completing the proof. QED.

**Theorem 1:** *Assume that all households in E have the same probability* $\Pi$ *over the distribution of risks in the population. Then any Arrow–Debreu equilibrium allocation* $(p^*,z^*)$ *of E (and therefore any Pareto optimum) can be achieved within the general equilibrium economy with incomplete markets* $E_I$ *by introducing a total of A mutual insurance contracts to hedge against individual risk, and A statistical securities to hedge against social risk. In a regular economy with two individual states and identical preferences, even if agents have different probabilities, there is always an Arrow–Debreu equilibrium* $(p^*,z^*)$ *in E which is achievable within the incomplete economy* $E_I$ *with the introduction of A mutual insurance contracts and A statistical securities.*

[11] This follows from Dierker (1982), p. 807, noting that his condition $D$ is implied by our assumption that preferences are strictly increasing (see Dierker's remark following the statement of property $D$ on p. 799).

*Proof:* Consider first the case where all households have the same probabilities, that is, $\Pi^h = \Pi^j = \Pi$. By Proposition 1, an Arrow–Debreu equilibrium of $E$ has the same prices $p_\sigma{}^* = p_r^*$ and the same consumption vectors $z_{h\sigma}^* = z_{hr}^*$ for each $h$, at each social state $\sigma$ with $r(\sigma) = r$. Define $\Omega(r)$ as the set of social states mapping to a given statistical state $r$, that is, $\Omega(r) = \{\sigma \in \Omega : r(\sigma) = r\}$. The budget constraint equation is

$$p^*\left(z_h^* - e_h\right) = \sum_\sigma p_\sigma^*\left(z_{h,\sigma}^* - e_{h\sigma}\right)$$
$$= \sum_r p_r^* \sum_{\sigma\in\Omega(r)}\left(z_{h\sigma}^* - e_{h\sigma}\right) = 0$$

Individual endowments depend on individual states and not on social states, so that $e_{h\sigma} = e_{hs(\sigma)} = e_{hs}$. Furthermore, by Proposition 1 equilibrium prices depend on $r$ and not on $\sigma$, so that for each $r$ the equilibrium consumption vector $z_{h\sigma}$ can be written as $z_{hs}$. The individual budget constraint is therefore $\Sigma_r p_r^* \Sigma_{s(r)}(z_{hs} - e_{hs})$, where summation over $s(r)$ indicates summation over all individual states $s$ that occur in any social state leading to $r$, that is, that are in the set $\Omega(r)$. Let $\#\Omega(r)$ be the number of social states in $\Omega(r)$. As $\Pi_{s|r} = r_s$ is the proportion of households in state $s$ within the statistical state $r$, we can finally rewrite the budget constraint equation (4.2) of the household $h$ as:

$$\#\Omega(r)\sum_r p_r^* \sum_s \#\Omega(r)\Pi_{s|r}\left(z_{hs} - e_{hs}\right) = 0 \qquad \text{(A.1)}$$

Using equation (3.1), the household's maximization problem can therefore be expressed as:

$$\max \sum_{s,r} \Pi_{sr} U^h\left(z_{hsr}\right) \text{ subject to } \left(\text{A.1}\right)$$

and the equilibrium allocation $z_h^*$ by definition solves this problem. Similarly, we may rewrite the market clearing condition (4.5) as follows:

$$\sum_h\left(z_h^* - e_h\right) = \sum_h\left(z_{h\sigma}^* - e_{hs(\sigma)}\right) = 0, \quad \forall\sigma\in\Omega$$

Rewriting the market clearing condition (4.3) in terms of statistical states $r$, and within each $r$, individual states $s$, we obtain

$$\sum_s r_s H\left(z_{hr}^* - e_{sr}^h\right) = 0, \quad \forall r \in R \tag{A.2}$$

or equivalently

$$\sum_s \Pi_{s|r} H\left(z_{hr}^* - e_{sr}^h\right) = 0, \quad \forall r \in R$$

Using these relations, we now show that any Arrow–Debreu equilibrium allocation $z^* = (z_{hr}^*)$ is within the budget constraints (5.2) of the economy $E_I$ for each $\sigma \in \Omega$, *provided that* for each $\sigma \in \Omega$ we add the income derived from a statistical security $A_r, r = r(\sigma)$, and, given $r(\sigma)$, the income derived from mutual or contingent insurance contracts $m_{sr}^h = m_{s(\sigma)r(\sigma)}^h$, $s = 1, \ldots, S$. We introduce $A$ statistical securities and $I.A$ mutual insurance contracts in the general equilibrium economy with incomplete markets $E_I$. The quantity of the security $A_r$ purchased by household $h$ in statistical state $r$, when equilibrium prices are $p^*$, is

$$a_r^{h*} = \sum_s \Pi_{s|r} p_r^*\left(z_{hr}^* - e_{hs}\right) \tag{A.3}$$

The quantity $a_r^{h*}$ has a very intuitive interpretation. It is the expected amount by which the value of equilibrium consumption exceeds the value of endowments, conditional on being in statistical state $r$. So *where the law of large numbers applies*, the statistical securities purchased deliver enough to balance a household's budget in each statistical state. Otherwise, differences between the average and each individual state are taken care of by the mutual insurance contracts. Note that equation (A.2) implies that the total amount of each security supplied is zero, that is, $\Sigma_h a_r^{h*} = 0$ for all $r$, so that this corresponds to the initial endowments of the incomplete economy $E_I$. Furthermore, $\Sigma_r a_r^{h*} = 0$ by (A.1), so that each household $h$ is within her/his budget in $E_I$.

We now introduce a mutual insurance contract as follows. The transfer made by individual $h$ in statistical state $r$ and individual state $s$, when prices are $p_r^*$, is

$$m_{sr}^{h*} = p_r^*\left(z_{hr}^* - e_{hr}\right) - a_r^{h*} \tag{A.4}$$

Note that, as remarked above, $m_{sr}^{h*}$ is just the difference between the actual income-expenditure gap, given that individual state $s$ is realized, and the expected income-expenditure gap $a_r^{h*}$ in statistical state $r$, which is covered by statistical securities. In each statistical state $r$, the sum over all $h$ and $s$ of all transfers $m_{sr}^{h*}$ equals zero, that is, the insurance premia match exactly the payments. For any given $r$,

$$\sum_{h,s} H\Pi_{s|r} m_{sr}^{h*} = \sum_{h,s} H\Pi_{s|r} p_r^*\left(z_{hr}^* - e_{hs}\right) - \sum_h H a_r^{h*} \sum_s \Pi_{s|r} = 0 \tag{A.5}$$

because $\Sigma_s \Pi_{s|r} = 1$. Therefore, the $\{m_{sr}^{h*}\}$ meet the definition of mutual insurance contracts. Finally, note that with $N$ spot markets, $A$ statistical securities $\{a_r\}$ and mutual insurance contracts $\{m_{sr}^h\}$

$$p_r^*\left(z_{hr}^* - e_s^h\right) = m_{sr}^{h*} + a_r^{h*},$$
$$\forall \sigma \in \Omega \text{ with } r(\sigma) = r, s = s(\sigma) \tag{A.6}$$

so that equation (5.2) is satisfied for each $\sigma \in \Omega$. This establishes that when all households have the same probabilities over social states, all Arrow–Debreu equilibrium allocation $z^*$ of $E$ can be achieved within the incomplete markets economy $E_I$ when $A$ securities and $A$ mutual insurance contracts are introduced into $E_I$, and completes the proof of the first part of the proposition dealing with common probabilites.

Consider now the case where the economy $E$ is regular, different households in $E$ have different probabilities over social states but have the same preferences, and $S = 2$. By Proposition 2, we know that within the set of equilibrium prices there is one $p^*$ in which at all social states $\sigma \in \Omega(r)$ for a given $r$, the equilibrium prices are the same, that is, $p_\sigma^* = p_r^*$. In particular, if $E$ has a unique equilibrium $(p^*,z^*)$, it must have this property. It follows from the above arguments that the equilibrium $(p^*,z^*)$ must maximize (3.1) subject to (A.1). Note, however, that now for the same $r$, $z_{hsr}^*$ may be different from $z_{hs'r}^*$ when $s \neq s'$. Now define the quantity of the security $A_r$ purchased by a household in the statistical state $r$ by

$$a_r^{h*} = \sum_s \Pi_{s|r}^h p_r^*\left(z_{hsr}^* - e_s^h\right) \tag{A.7}$$

and the mutual insurance transfer made by a household in statistical state $r$ and individual state $s$, by

$$m_{sr}^{h*} = p_r^*\left(z_{hsr}^* - e_s^h\right) - a_r^{h*} \tag{A.8}$$

As before, $\Sigma_r a_r^{h*} = 0$ and for any given $r$, $\Sigma_{h,s}\Pi^h_{s|r} Hm^{h*}_{sr} = \Sigma_{h,s} r_s Hm^{h*}_{sr} = 0$, so that the securities purchased correspond to the initial endowments of the economy $E_I$ and at any statistical state the sum of the premia and the sum of the payments of the mutual insurance contracts match, completing the proof. QED.

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