Sustainability: Dynamics and Uncertainty

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2.1. Sustainable Use of Renewable Resources *

1. Introduction

We consider here optimal use patterns for renewable resources. Many important resources are in this category: obvious ones are fisheries and forests. Soils, clean water, landscapes, and the capacities of ecosystems to assimilate and degrade wastes are other less obvious examples.[1] All of these have the capacity to renew themselves, but in addition all can be overused to the point where they are irreversibly damaged. Picking a time-path for the use of such resources is clearly important: indeed, it seems to lie at the heart of any concept of sustainable economic management.

We address the problem of optimal use of renewable resources under a variety of assumptions both about the nature of the economy in which these resources are embedded and about the objective of that economy. In this second respect, we are particularly interested in investigating the consequences of a definition of sustainability as a form of intertemporal optimality recently introduced by Chichilnisky [7], and comparing these consequences with those arising from earlier definitions of intertemporal optimality. In terms of the structure of the economy considered, we review the problem initially in the context of a model where a renewable resource is the only good in the economy, and then subsequently we extend the analysis to include the accumulation of capital and the existence of a productive sector to which the resource is an input.

Although we focus here on the technical economic issues of defining and characterizing paths which are optimal, in various senses, in the presence of renewable resources, one should not loose sight of the very real motivation underlying these exercises: many of the earth's most important biological and ecological resources are renewable, so that in their management we confront

* This paper draws heavily on earlier research by one or more of the three authors, namely Beltratti, Chichilnisky and Heal [1–3], Chichilnisky [7, 8], and in particular many of the results here were presented in Heal [18].

G. Chichilnisky et al. (eds.), Sustainability: Dynamics and Uncertainty, 49–76.

the fundamental choice which underlies this paper, namely their extinction, or their preservation as viable species. In this context the recent discussion of sustainability or sustainable management of the earth's resources is closely related to the issues of concern to us. (For a more comprehensive discussion of issues relating to sustainability and its interpretation in economic terms, see [18]. For a review of the basic theory of optimal intertemporal use of resources, see [10, 11, 15].)

We assume, as in [19] and in earlier work by some or all of us [1–3] that the renewable resource is valued not only as a source of consumption but also as a source of utility in its own right: this means that the existing stock of the resource is an argument of the utility function. The instantaneous utility function is therefore $u(c, s)$, where c is consumption and s the remaining stock of the resource. This is clearly the case for forests, which can be used to generate a flow of consumption via timber, and whose stock is a source of pleasure. Similarly, it is true for fisheries, for landscapes, and probably for many more resources. Indeed, in so far as we are dealing with a living entity, there is a moral argument, which we will not evaluate here, that we should value the stock to attribute importance to its existence in its own right and not just instrumentally as a source of consumption.

2. The Utilitarian Case without Production

We begin by considering the simplest case, that of a conventional utilitarian objective with no production: the resource is the only good in the economy. For this framework we characterize the utilitarian optimum, and then extend these results to other frameworks. The maximand is the discounted integral of utilities from consumption and from the existence of a stock, $\int_0^\infty u(c, s)\, e^{-\delta t} dt$, where $\delta > 0$ is a discount rate. As the resource is renewable, its dynamics are described by

$$\dot{s}_t = r(s_t) - c_t.$$

Here r is the growth rate of the resource, assumed to depend only on its current stock. More complex models are of course possible, in which several such systems interact: a well-known example is the predator-prey system. In general, r is a concave function which attains a maximum at a finite value of s, and declines thereafter. This formulation has a long and classical history, which is reviewed in [11]. In the field of population biology, $r(s_t)$ is often taken to be quadratic, in which case an unexploited population (i.e., $c_t = 0\, \forall t$) grows logistically. Here we assume that $r(0) = 0$, that there exists a positive stock level $\overline{s}$ at which $r(\overline{s}) = 0\ \forall s \geq \overline{s}$, and that $r(s)$ is strictly concave and twice continuously differentiable for $s \in (0, \overline{s})$. The overall problem can now be specified as

$$\max \int_0^\infty u(c, s)\, e^{-\delta t} dt \text{ s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given.} \tag{1}$$

The Hamiltonian in this case

$H = u(c_t, s_t)\, e^{-\delta t} +$

Maximization with respect tc
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$u_c(c_t, s_t) = \lambda_t$

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2.1. *Stationary Solutions*

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$r(s_t) = c_t$

$\frac{u_2'(s_t)}{u_1'(c_t)} = \delta - r'(s_t)$

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[17, 18].

The Hamiltonian in this case is

$$H = u\left(c_t, s_t\right) e^{-\delta t} + \lambda_t e^{-\delta t}\left[r\left(s_t\right) - c_t\right].$$

Maximization with respect to consumption gives as usual the equality of the marginal utility of consumption to the shadow price for positive consumption levels:

$$u_c\left(c_t, s_t\right) = \lambda_t$$

and the rate of change of the shadow price is determined by

$$\frac{d}{dt}\left(\lambda_t e^{-\delta t}\right) = -\left[u_s\left(c_t, s_t\right) e^{-\delta t} + \lambda_t e^{-\delta t} r'\left(s_t\right)\right].$$

To simplify matters we shall take the utility function to be separable in c and s: $u(c, s) = u_1(c) + u_2(s)$, each taken to be strictly concave and twice differentiable. In this case a solution to the problem (1) is characterized by

$$\left.\begin{aligned} u_1'\left(c_t\right) &= \lambda_t \\ \dot{s}_t &= r\left(s_t\right) - c_t \\ \dot{\lambda}_t - \delta\lambda_t &= -u_2'\left(s_t\right) - \lambda_t r'\left(s_t\right) \end{aligned}\right\}. \tag{2}$$

In studying these equations, we first analyze their stationary solution, and then examine the dynamics of this system away from the stationary solution.

2.1. *Stationary Solutions*

At a stationary solution, by definition s is constant so that $r\left(s_t\right) = c_t$: in addition, the shadow price is constant so that

$$\delta u_1'\left(c_t\right) = u_2'\left(s_t\right) + u_1'\left(c_t\right) r'\left(s_t\right).$$

Hence:

PROPOSITION 1. *A stationary solution to the utilitarian optimal use pattern (2) satisfies*

$$\left.\begin{aligned} r\left(s_t\right) &= c_t \\ \frac{u_2'\left(s_t\right)}{u_1'\left(c_t\right)} &= \delta - r'\left(s_t\right) \end{aligned}\right\}. \tag{3}$$

The first equation in (3) just tells us that a stationary solution must lie on the curve on which consumption of the resource equals its renewal rate: this is obviously a prerequisite for a stationary stock. The second gives us a relationship between the slope of an indifference curve in the c–s plane and the slope of the renewal function at a stationary solution: the indifference curve cuts the renewal function from above. Such a configuration is shown in Figure 1. This is just the result that the slope of an indifference curve should equal the discount rate if $r'(s) = 0\ \forall s$, i.e., if the resource is non-renewable [17, 18].

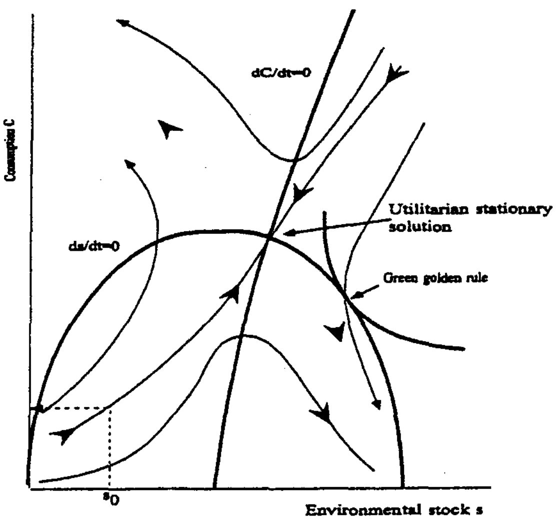

Figure 1. Dynamics of the utilitarian solution.

There is a straightforward intuitive interpretation to the second equation in (3). Consider reducing consumption by an amount Δc and increasing the stock by the same amount. The welfare loss is $\Delta c u_1'$: there is a gain from increasing the stock of $\Delta c u_2'$, which continues for ever, so that we have to compute its present value. But we also have to recognize that the increment to the stock will grow at the rate r': hence the gain from the increase in stock is the present value of an increment which compounds at rate r'. Hence the total gain is

$$\Delta c \int_0^{\infty} u_2' e^{r't} e^{-\delta t} dt = u_2' \Delta c / (r' - \delta).$$

When gains and losses just balance out, we have

$$u_1' + u_2' / (r' - \delta) = 0$$

which is just the second equation of (3). So (3) is a very natural and intuitive characterization of optimality.

2.2. *Dynamic Behavior*

What are the dynamics of this system outside of a stationary solution? These are also shown in Figure 1. They are derived by noting the following facts:

1. beneath the curve $r(s) = c$, s is rising as consumption is less than the growth of the resource.
2. above the curve $r(s) = c$, s is falling as consumption is greater than the growth of the resource.

3. on the curve $r(s) = c$, s is
4. from (2), the rate of chang

$$u_1''(c)\dot{c} = u_1'(c)\left[\delta - r'\right.$$

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$$\dot{s}_t = r(s_t) \cdot$$

around the stationary solut point. The determinant of

$$r'(s)\{\delta - r'(s)\} - \frac{1}{u_1''}$$

which is negative for any sustainable yield.

Hence the dynamics of path mality are as shown in figure 1

PROPOSITION 2.[2] *For small the derivatives r', r'' or u_1', al tend to the stationary solution order conditions (2), and follo Figure 1 leading to the stationa s_0, there is a corresponding v of the stable branches leading stationary solution depends on of the stationary stock as this tends to a point satisfying $u_2'/$ an indifference curve of $u(c, s$ graph of the renewal function.*

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Note that if the initial resou *consumption, stock and utility* that because the resource is re

3. on the curve $r(s) = c$, s is constant.
4. from (2), the rate of change of c is given by

$$u_1''(c)\,\dot{c} = u_1'(c)\,[\delta - r'(s)] - u_2'(s)\,.$$

The first term here is negative for small s and vice versa: the second is negative and large for small s and negative and small for large s. Hence c is rising for small s and vice versa: its rate of change is zero precisely when the rate of change of the shadow price is zero, which is on a line of positive slope containing the stationary solution.

5. by linearizing the system

$$\left.\begin{aligned} u_1''(c)\,\dot{c} &= u_1'(c)\,[\delta - r'(s)] - u_2'(s) \\ \dot{s}_t &= r(s_t) - c_t \end{aligned}\right\}$$

around the stationary solution, one can show that this solution is a saddle point. The determinant of the matrix of the linearized system is

$$r'(s)\{\delta - r'(s)\} - \frac{1}{u_1''}\{u_1'r'' + u_2''\}$$

which is negative for any stationary stock in excess of the maximum sustainable yield.

Hence the dynamics of paths satisfying the necessary conditions for optimality are as shown in figure 1, and we can establish the following result:

PROPOSITION 2.[2] *For small values of the discount rate δ or large values of the derivatives r', r'' or u_1', all optimal paths for the utilitarian problem (1) tend to the stationary solution (3). They do so along a path satisfying the first order conditions (2), and follow one of the two branches of the stable path in Figure 1 leading to the stationary solution. Given any initial value of the stock s_0, there is a corresponding value of c_0 which will place the system on one of the stable branches leading to the stationary solution. The position of the stationary solution depends on the discount rate, and moves to higher values of the stationary stock as this decreases. As $\delta \to 0$, the stationary solution tends to a point satisfying $u_2'/u_1' = r'$, which means in geometric terms that an indifference curve of $u(c, s)$ is tangent to the curve $c = r(s)$ given by the graph of the renewal function.*

This result characterizes optimal paths for the problem (1). It does not prove the existence of such paths. The Appendix gives an argument which establishes that an optimal path exists for all of the problems whose solutions are characterized in this paper.

Note that if the initial resource stock is low, *the optimal policy requires that consumption, stock and utility all rise monotonically over time.* The point is that because the resource is renewable, both stocks and flows can be built up over time provided that consumption is less than the rate of regeneration, i.e., the system is inside the curve given by the graph of the renewal function $r(s)$. In practice, unfortunately, many renewable resources are being consumed at a rate greatly in excess of their rates of regeneration: in terms of Figure 1, the current consumption rate c_t is much greater than $r(s_t)$. So taking advantage of the regeneration possibilities of these resources would in many cases require sharp limitation of current consumption. Fisheries are a widely-publicized example: another is tropical hardwoods and tropical forests in general. Soil is a more subtle example: there are processes which renew soil, so that even if it suffers a certain amount of erosion or of depletion of its valuable components, it can be replaced. But typically human use of soils is depleting them at rates far in excess of their replenishment rates.

Proposition 2 gives conditions necessary for a path to be optimal from problem (1). Given the concavity of $u(c, s)$ and of $r(s)$, one can invoke standard arguments to show that these conditions are also sufficient (see, for example, [22]).

3. Renewable Resources and the Green Golden Rule

We can use the renewable framework to ask the question: what configuration of the economy gives the maximum sustainable utility level?[3] There is a simple answer.

First, note that a sustainable utility level must be associated with a sustainable configuration of the economy, i.e., with sustainable values of consumption and of the stock. But these are precisely the values that satisfy the equation

$$c_t = r(s_t)$$

for these are the values which are feasible and at which the stock and the consumption levels are constant. Hence in Figure 1, we are looking for values which lie on the curve $c_t = r(s_t)$. Of these values, we need the one which lies on the highest indifference curve of the utility function $u(c, s)$: this point of tangency is shown in the figure. At this point, the slope of an indifference curve equals that of the renewal function, so that the marginal rate of substitution between stock and flow equals the marginal rate of transformation along the curve $r(s)$. Hence:

PROPOSITION 3.[4] *The maximum sustainable utility level (the green golden rule) satisfies*

$$\frac{u_2'(s_t)}{u_1'(c_t)} = -r'(s_t).$$

Recall from (3) that as the discount rate goes to zero, the stationary solution to the utilitarian case tends to such a point. Note also that any path

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3.1. *Ecological Stability*

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which approaches the tangency of an indifference curve with the reproduction function, is optimal according to the criterion of maximizing sustainable or long-run utility. In other words, this criterion of optimality only determines the limiting behavior of the economy: it does not determine how the limit is approached. This clearly is a weakness: of the many paths which approach the green golden rule, some will accumulate far more utility than others. One would like to know which of these is the best, or indeed whether there is such a best. It transpires that in general there is not. We return to this later.

3.1. *Ecological Stability*

An interesting fact is that the green golden rule, and also for low enough discount rates the utilitarian solution, require stocks of the resource which are in excess of that giving the maximum sustainable yield, which is of course the stock at which the maximum of $r(s)$ occurs. This is important because only resource stocks in excess of that giving the maximum sustainable yield are stable under the natural population dynamics of the resource [21]: they are ecologically stable. To see this, consider a fixed depletion rate d, so that the resource dynamics is just

$$\dot{s} = r(s) - d.$$

For $d < \max_s r(s)$, there are two values of s which give stationary solutions to this equation, as shown in Figure 2. Call the smaller s_1 and the larger s_2. Clearly for $s > s_2$, $\dot{s} < 0$, for $s_1 < s < s_2, \dot{s} > 0$, and for $s < s_1$, $\dot{s} < 0$, as shown in Figure 2. Only the stock to the right of the maximum sustainable yield is stable under the natural population adjustment process: high discount rates, and utilitarian optimal policies when the stock of the resource is not an argument of the utility function, will give stationary stocks below the maximum sustainable yield.

4. The Rawlsian Solution

Consider the initial stock level s_1 in Figure 1: the utilitarian optimum from this is to follow the path that leads to the saddle point. In this case, as noted, consumption, stock and utility are all increasing. So the generation which is least well off, is the first generation. What is the Rawlsian solution in the present model, with initial stock s_1? It is easy to verify that this involves setting $c = r(s_1)$ for ever: this gives a constant utility level, and gives the highest utility level for the first generation compatible with subsequent levels being no lower. This remains true for any initial stock no greater than that associated with the green golden rule: for larger initial stocks, the green golden rule is a Rawlsian optimum. Formally,

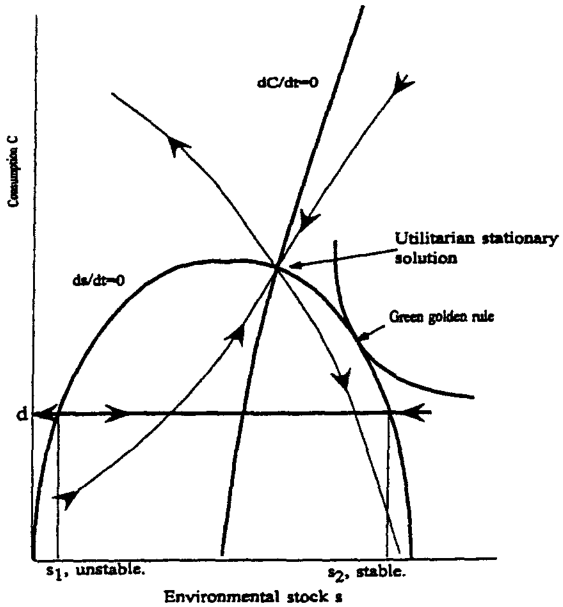

Figure 2. The dynamics of the renewable resource under a constant depletion rate.

PROPOSITION 4. *For an initial resource stock s_1 less than or equal to that associated with the green golden rule, the Rawlsian optimum involves setting $c = r(s_1)$ for ever. For s_1 greater than the green golden rule stock, the green golden rule is a Rawlsian optimum.*

5. Chichilnisky's Criterion

Next, we ask how the Chichilnisky criterion [7, 8] alters matters when applied to an analysis of the optimal management of renewable resources. Recall that Chichilnisky's criterion ranks paths according to the sum of two terms, one an integral of utilities against a finite countably additive measure and one a purely finitely additive measure defined on the utility stream of the path. The former is just a generalization of the discounted integral of utilities (generalized in the sense that the finite countably additive measure need not be an exponential discount factor). The latter term can be interpreted as a *sustainable utility level*: Chichilnisky shows that any ranking of intertemporal paths which satisfies certain basic axioms must be representable in this way. The problem now is to pick paths of consumption and resource accumulation over time to:

$$\max \alpha \int_0^{\infty} u(c_t, s_t) \ldots$$

s.t. $\dot{s}_t = \ldots$

where $f(t)$ is a finite countab

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$$u_1''(c)\dot{c} = u_1'(c)[\delta$$

$$\dot{s}_t = r(s_t) -$$

Denote by v_0 the 2-vector c

$\{\bar{c}_t, \bar{s}_t\}(v_0)$. Follow this patl

the green golden rule, i.e.,

$\bar{s}_{t'} = s^*$, and then at $t = t'$

to the green golden rule, i.

because $c_t < r(s_t)$ along

Formally, this path is $(c_t, s$

$\bar{s}_{t'} = s^*$, and $c_t = r(s^*)$, s_t

Any such path will satisfy

up to time t' and will leac

therefore attain a maximum

However, the utility integra

$$\left.\begin{array}{l}\max \alpha \displaystyle\int_0^\infty u(c_t, s_t) f(t)\, dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t) \\ \text{s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given.}\end{array}\right\}, \tag{4}$$

where $f(t)$ is a finite countably additive measure.

The change in optimal policy resulting from the change in the criterion of optimality is quite dramatic. With the Chichilnisky criterion and the measure $f(t)$ given by an exponential discount factor, i.e., $f(t) = e^{-\delta t}$, *there is no solution to the overall optimization problem.*[5] There is a solution only if $f(t)$ takes a different, non-exponential form, implying a non-constant discount rate which tends asymptotically to zero. Chichilnisky's criterion thus links in an unexpected way with recent discussions of individual attitudes towards the future: there is empirical evidence that individuals making intertemporal choices act as if they have non-constant discount rates which decline over time. Formally:

PROPOSITION 5.[6] *The problem (4) has no solution, i.e., there is no optimal pattern of use of a renewable resource using the Chichilnisky criterion with a constant discount rate.*

Proof. Consider first the problem

$$\max \int_0^\infty u(c_t, s_t) e^{-\delta t} dt \text{ s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given.}$$

The dynamics of the solution is shown in Figure 1, reproduced here as Figure 3.

It differs from the problem under consideration by the lack of the term in limiting utility in the maximand. Suppose that the initial stock is s_0 in Figure 3. Pick an initial value of c, say c_0, below the path leading to the saddle-point, and follow the path from c_0 satisfying the utilitarian necessary conditions given above:

$$\begin{aligned} u_1''(c)\dot{c} &= u_1'(c)\left[\delta - r'(s)\right] - u_2'(s), \\ \dot{s}_t &= r(s_t) - c_t. \end{aligned}$$

Denote by v_0 the 2-vector of initial conditions: $v_0 = (c_0, s_0)$. Call this path $\{\bar{c}_t, \bar{s}_t\}(v_0)$. Follow this path until it leads to the resource stock corresponding the green golden rule, i.e., until the t' such that on the path $\{\bar{c}_t, \bar{s}_t\}(v_0)$, $\bar{s}_{t'} = s^*$, and then at $t = t'$ increase consumption to the level corresponding to the green golden rule, i.e., set $c_t = r(s^*)$ for all $t \geq t'$. This is feasible because $c_t < r(s_t)$ along such a path. Such a path is shown in Figure 3. Formally, this path is $(c_t, s_t) = \{\bar{c}_t, \bar{s}_t\}(v_0)\ \forall t \leq t'$ where t' is defined by $\bar{s}_{t'} = s^*$, and $c_t = r(s^*)$, $s_t = s^*\ \forall t > t'$.

Any such path will satisfy the necessary conditions for utilitarian optimality up to time t' and will lead to the green golden rule in finite time. It will therefore attain a maximum of the term $\lim_{t\to\infty} u(c_t, s_t)$ over feasible paths. However, the utility integral which constitutes the first part of the maximand

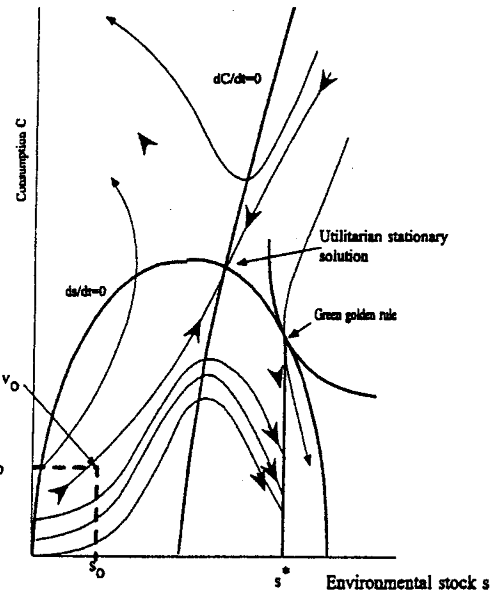

Figure 3. A sequence of consumption paths with initial stock s_0 and initial consumption level below that leading to the utilitarian stationary solution and converging to it. Once the stock reaches s^* consumption is set equal to $r(s^*)$. The limit is a path which approaches the utilitarian stationary solution and not the green golden rule.

can be improved by picking a slightly higher initial value c_0 for consumption, again following the first order conditions for optimality and reaching the green golden rule slightly later than t'. This does not detract from the second term in the maximand. By this process it will be possible to increase the integral term in the maximand without reducing the limiting term and thus to approximate the independent maximization of both terms in the maximand: the discounted utilitarian term, by staying long enough close to the stable manifold leading to the utilitarian stationary solution, and the limit (purely finitely additive) term by moving to the green golden rule very far into the future.

Although it is possible to *approximate* the maximization of both terms in the maximand independently by postponing further and further the jump to the green golden rule, there is no feasible path that actually *achieves* this maximum. The supremum of the values of the maximand over feasible paths is approximated arbitrarily closely by paths which reach the green golden rule at later and later dates, but the limit of these paths never reaches the green golden rule and so does not achieve the supremum. More formally, consider the limit of paths $(c_t, s_t) = \{\overline{c}_t, \overline{s}_t\}(v_0)\ \forall t \le t'$ where t' is defined by $\overline{s}_{t'} = s^*$, and $c_t = r(s^*), s_t = s^*\ \forall t > t'$ as c_0 approaches the stable manifold of the utilitarian optimal solution. On this limiting path $s_t < s^*\ \forall t$.

Hence there is no solution to

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5.1. *Declining Discount Rate*

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$$\alpha \int_0^\infty u(c_t, s_t)\, \Delta(t) \ldots$$

where $\Delta(t)$ is the discount fa
rate $q(t)$ at time t is the prop

$$q(t) = -\frac{\dot{\Delta}(t)}{\Delta(t)}$$

and we assume that the disco

$$\lim_{t \to \infty} q(t) = 0.$$

So the overall problem is now

$$\max \alpha \int_0^\infty u(c_t, s_t) \ldots$$
$$\text{s.t. } \dot{s}_t = \ldots$$

where the discount factor Δ
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solution:[7] in fact, it is the solu
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Formally,

Hence there is no solution to (4). □

Intuitively, the non-existence problem arises here because it is always possible to postpone further into the future moving to the green golden rule, with no cost in terms of limiting utility values but with a gain in terms of the integral of utilities. This is possible because of the renewability of the resource. There is no equivalent phenomenon for an exhaustible resource [18].

5.1. *Declining Discount Rates*

With the Chichilnisky criterion formulated as

$$\alpha \int_0^\infty u(c_t, s_t)\, e^{-\delta t} dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t),$$

there is no solution to the problem of optimal management of a renewable resource. In fact as noted the discount factor does not have to be an exponential function of time. The criterion can be stated slightly differently, in a way which is still consistent with Chichilnisky's axioms and which is also consistent with solving the renewable resource problem. This reformulation builds on a point that we have noted before, namely that for the discounted utilitarian case, as the discount rate goes to zero, the stationary solution goes to the green golden rule. We shall, therefore, consider a modified objective function

$$\alpha \int_0^\infty u(c_t, s_t)\, \Delta(t)\, dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t),$$

where $\Delta(t)$ is the discount factor at time t, $\int_0^\infty \Delta(t)\, dt$ is finite, the discount rate $q(t)$ at time t is the proportional rate of change of the discount factor:

$$q(t) = -\frac{\dot{\Delta}(t)}{\Delta(t)}$$

and we assume that the discount rate goes to zero with t in the limit:

$$\lim_{t\to\infty} q(t) = 0. \tag{5}$$

So the overall problem is now

$$\max \alpha \int_0^\infty u(c_t, s_t)\, \Delta(t)\, dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t)$$
$$\text{s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given,}$$

where the discount factor $\Delta(t)$ satisfies the condition (5) that the discount rate goes to zero in the limit. We will show that for this problem, there is a solution:[7] in fact, it is the solution to the utilitarian problem of maximizing just the first term in the above maximand, $\int_0^\infty u(c_t, s_t)\, \Delta(t)\, dt$. As before we take the utility function to be separable in its arguments: $u(c, s) = u_1(c) + u_2(s)$. Formally, PROPOSITION 6.[8] *Consider the problem*

$$\max \alpha \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\,dt + (1-\alpha)\lim_{t\to\infty}\{u_1(c) + u_2(s)\},$$
$$0 < \alpha < 1, \ \textit{s.t. } \dot{s}_t = r(s_t) - c_t, s_0 \textit{ given},$$

where $q(t) = -(\dot{\Delta}(t)/\Delta(t))$ *and* $\lim_{t\to\infty} q(t) = 0$. *A solution to this problem is identical to the solution of* "$\max \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\,dt$ *subject to the same constraint". In words, the conditions characterizing a solution to the utilitarian problem with the variable discount rate which goes to zero also characterize a solution to the overall problem.*

Proof. Consider first the problem $\max \alpha \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\,dt$ s.t. $\dot{s}_t = r(s_t) - c_t$, s_0 given. We shall show that any solution to this problem approaches and attains the green golden rule asymptotically, which is the configuration of the economy which gives the maximum of the term $(1 - \alpha)\lim_{t\to\infty} u(c_t, s_t)$. Hence this solution solves the overall problem. The Hamiltonian for the integral problem is now

$$H = \{u_1(c) + u_2(s)\}\Delta(t) + \lambda_t\Delta(t)[r(s_t) - c_t]$$

and maximization with respect to consumption gives as before

$$u_1'(c_t) = \lambda_t.$$

The rate of change of the shadow price λ_t is determined by

$$\frac{d}{dt}(\lambda_t\Delta(t)) = -\left[u_2'(s_t)\Delta(t) + \lambda_t\Delta(t)r'(s_t)\right].$$

The rate of change of the shadow price is, therefore,

$$\dot{\lambda}_t\Delta(t) + \lambda_t\dot{\Delta}(t) = -u_2'(s_t)\Delta(t) - \lambda_t\Delta(t)r'(s_t). \tag{6}$$

As $\dot{\Delta}(t)$ depends on time, this equation is not autonomous, i.e., time appears explicitly as a variable. For such an equation, we cannot use the phase portraits and associated linearization techniques used before, because the rates of change of c and s depend not only on the point in the c–s plane but also on the date. Rearranging and noting that $\dot{\Delta}(t)/\Delta(t) = q(t)$, we have

$$\dot{\lambda}_t + \lambda_t q(t) = -u_2'(s_t) - u_1'(c_t)r'(s_t).$$

But in the limit $q = 0$, so in the limit this equation is autonomous: this equation and the stock growth equation form what has recently been called in dynamical systems theory an asymptotically autonomous system [4]. According to proposition 1.2 of [4], the asymptotic phase portrait of this non-autonomous system

$$\left.\begin{aligned} \dot{\lambda}_t + \lambda_t q(t) &= -u_2'(s_t) - u_1'(c_t)r'(s_t) \\ \dot{s}_t &= r(s_t) - c_t \end{aligned}\right\} \tag{7}$$

is the same as that of the a

$$\dot{\lambda}_t = -u_2'(s_t) -$$
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PROPOSITION 7.[10] *Cor*

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is the same as that of the autonomous system

$$\left.\begin{aligned} \dot{\lambda}_t &= -u_2'(s_t) - u_1'(c_t)\, r'(s_t) \\ \dot{s}_t &= r(s_t) - c_t \end{aligned}\right\} \tag{8}$$

which differs only in that the non-autonomous term $q(t)$ has been set equal to zero.[9] The pair of Equations (8) is an autonomous system and the asymptotic stability properties of original system (7) will be the same as those of the associated limiting autonomous system (8). This latter system can be analyzed by the standard techniques used before. At a stationary solution of (8), $\dot{\lambda}_t = 0$ and $c_t = r(s_t)$, so that

$$\frac{u_2'}{u_1'} = -r' \quad \text{and} \quad c_t = r(s_t)$$

which is just the definition of the green golden rule. Furthermore, by the arguments used above we can establish that the green golden rule is a saddlepoint of the system (8), as shown in Figure 3. So the optimal path for the problem

$$\text{“maximize } \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\, dt$$
$$\text{subject to } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given”}$$

is for any given initial stock s_0 to select an initial consumption level c_0 such that (c_0, s_0) is on the stable path of the saddle point configuration which approaches the Green Golden Rule asymptotically. But this path also leads to the maximum possible value of the term $\lim_{t\to\infty}\{u_1(c) + u_2(s)\}$, and therefore leads to a solution to the overall maximization problem. □

Figure 4 shows the behavior of an optimal path in this case. Intuitively, one can see what drives this result. The non-existence of an optimal path with a constant discount rate arose from a conflict between the long-run behavior of the path that maximizes the integral of discounted utilities, and that of the path that maximizes the long-run utility level. When the discount rate goes to zero in the limit, that conflict is resolved. In fact, one can show that it is resolved only in this case, as stated by the following proposition.

PROPOSITION 7.[10] *Consider the problem*

$$\max \alpha \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\, dt + (1-\alpha)\lim_{t\to\infty}\{u_1(c) + u_2(s)\},$$
$$0 < \alpha < 1,\ \textit{s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \textit{ given},$$

where $q(t) = -(\dot{\Delta}(t)/\Delta(t))$. *This problem has a solution only if* $\lim_{t\to\infty} q(t) = 0$. *In this case, the solution is characterized by the conditions which characterize the solution to* "max $\int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\, dt$ *subject to the same constraint*".

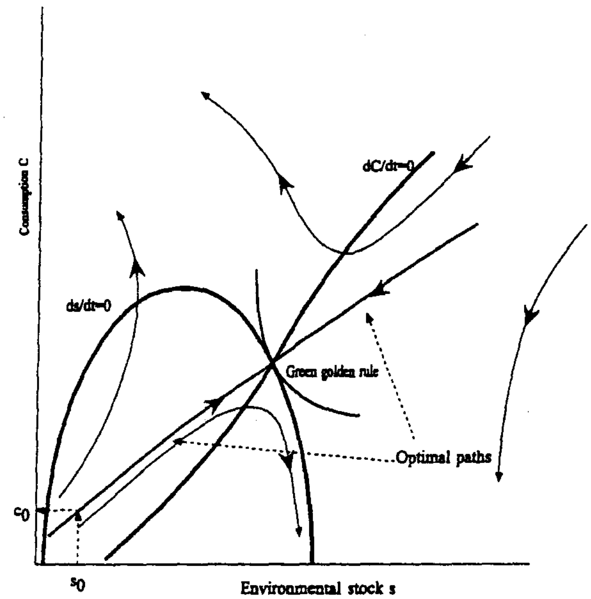

Figure 4. Asymptotic dynamics of the utilitarian solution for the case in which the discount rate falls to zero.

Proof. The "if" part of this was proven in the previous proposition, Proposition 6. The "only if" part can be proven by an extension of the arguments in Proposition 5, which established the non-existence of solutions in the case of a constant discount rate. To apply the arguments there, assume contrary to the proposition that $\liminf_{t\to\infty} q(t) = \overline{q} > 0$, and then apply the arguments of Proposition 5. $\square$

Existence of a solution to this problem is established in the Appendix.

5.2. *Examples*

To complete this discussion, we review some examples of discount factors which satisfy the condition that the limiting discount rate goes to zero. The most obvious is

$$\Delta(t) = e^{-\delta(t)t}, \quad \text{with} \quad \lim_{t\to\infty} \delta(t) = 0.$$

Another example[11] is

$$\Delta(t) = t^{-\alpha}, \quad \alpha > 1.$$

Taking the starting date to be $t = 1$,[12] we have

$$\int_1^\infty t^{-\alpha} dt = \frac{1}{\alpha - 1}$$

and

$$\frac{\dot{\Delta}}{\Delta} = \frac{-\alpha}{t} \to 0 \quad \text{as}$$

5.3. *Empirical Evidence on D*

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$$\frac{\dot{\Delta}}{\Delta} = \frac{-\alpha}{t} \to 0 \quad \text{as} \quad t \to \infty.$$

5.3. *Empirical Evidence on Declining Discount Rates*

Proposition 7 has substantial implications. It says that when we seek optimality with a criterion sensitive to the present and the long-run future, then with non-renewable resources existence of a solution is tied to the limiting behavior of the discount rate: in the limit, we have to treat present and future utilities symmetrically in the evaluation of the integral of utilities. In a certain sense, the treatment of present and future in the integral has to be made consistent with the presence of the term $\lim_{t\to\infty}\{u_1(c) + u_2(s)\}$ which places positive weight on the very long run.

There is a growing body of empirical evidence that people actually behave like this in evaluating the future (see, for example, [20]; see also [18] for a more comprehensive discussion). The evidence suggests that the discount rate which people apply to future projects depends upon, and declines with, the futurity of the project. Over relatively short periods up to perhaps five years, they use discount rates which are higher even than commercial rates – in the region of 15% or more. For projects extending about ten years, the implied discount rates are closer to standard rates – perhaps 10%. As the horizon extends the implied discount rates drops, to in the region of 5% for 30 to 50 years and down to of the order of 2% for 100 years. It is of great interest that a framework for intertemporal optimization that is sensitive to both present and future generates an implication for discounting that may rationalize a form of personal behavior that hitherto has been found irrational.

This empirically-identified behavior is consistent with results from natural sciences which find that human responses to a change in a stimulus are nonlinear, and are inversely proportional to the existing level of the stimulus. This is an example of the Weber–Fechner law, which is formalized in the statement that human response to a change in a stimulus is inversely proportional to the pre-existing stimulus. In symbols,

$$\frac{dr}{ds} = \frac{K}{s} \quad \text{or} \quad r = K \log s,$$

where r is a response, s a stimulus and K a constant. This has been found to apply to human responses to the intensity of both light and sound signals. We noted that the empirical results on discounting cited above suggest that something similar is happening in human responses to changes in the futurity of an event: a given change in futurity (e.g., postponement by one year) leads to a smaller response in terms of the decrease in weighting, the further the event already is in the future. In this case, the Weber–Fechner law can be applied to responses to distance in time, as well as to sound and light intensity, with the result that the discount rate is inversely proportional to distance into the future. Recalling that the discount factor is $\Delta(t)$ and the discount rate $q(t) = -\dot{\Delta}(t)/\Delta(t)$, we can formalize this as

$$q(t) = \frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{K}{t} \quad \text{or} \quad \Delta(t) = e^{K\log t} = t^K$$

for K a positive constant. Such a discount factor can meet all of the conditions we required above: the discount rate q goes to zero in the limit, the discount factor $\Delta(t)$ goes to zero and the integral $\int_1^\infty \Delta(t)\,dt = \int_1^\infty e^{K\log t}dt = \int_1^\infty t^K\,dt$ converges for K positive, as it always is. In fact, this interpretation gives rise to the second example of a non-constant discount rate considered in the previous section. A discount factor $\Delta(t) = e^{K\log t}$ has an interesting interpretation: the replacement of t by $\log t$ implies that we are measuring time differently, i.e. by equal proportional increments rather than by equal absolute increments.

5.4. *Time Consistency*

An issue which is raised by the previous propositions is that of *time consistency*. Consider a solution to an intertemporal optimization problem which is computed today and is to be carried out over some future period of time starting today. Suppose that the agent formulating it – an individual or a society – may at a future date recompute an optimal plan, using the same objective and the same constraints as initially but with initial conditions and starting date corresponding to those obtaining when the recomputation is done. Then we say that the initial solution is *time consistent* if this leads the agent to continue with the implementation of the initial solution. Another way of saying this is that a plan is time consistent if the passage of time alone gives no reason to change it. The important point is that the solution to the problem of optimal management of the renewable resource with a time-varying discount rate, stated in Proposition 7, is not time-consistent. A formal definition of time consistency is:[13]

DEFINITION 8. Let $(c_t^*, s_t^*)_{t=0,\infty}$ be the solution to the problem

$$\max \alpha \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\,dt + (1-\alpha)\lim_{t\to\infty}\{u_1(c) + u_2(s)\},$$
$$0 < \alpha < 1, \text{ s.t. } \dot{s}_t = r(s_t) - c_t, s_0 \text{ given.}$$

Let $(\widehat{c}_t, \widehat{s}_t)_{t=T,\infty}$ be the solution to the problem of optimizing from T on, given that the path $(c_t^*, s_t^*)_{t=0,\infty}$ has been followed up to date T., i.e., $(\widehat{c}_t, \widehat{s}_t)_{t=T,\infty}$ solves

$$\max \alpha \int_T^\infty \{u_1(c) + u_2(s)\}\Delta(t-T)\,dt + (1-\alpha)\lim_{t\to\infty}\{u_1(c) + u_2(s)\},$$
$$0 < \alpha < 1, \text{s.t. } \dot{s}_t = r(s_t) - c_t,\ s_T^* \text{ given.}$$

Then the original problem so
$(\widehat{c}_t, \widehat{s}_t)_{t=T,\infty} = (c_t^*, s_t^*)_{t=T,\infty}$
period $[T, \infty]$ is also a solutio
stock s_T^*, for any T.

It is shown in [14] that the sc
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PROPOSITION 9.[14] *The solu*
renewable resource with a dis
time consistent, i.e., the soluti

$$\max \alpha \int_0^\infty \{u_1(c) + u_2($$
$$0 < \alpha < 1, \text{ s.t. } \dot{s}_t = r(s_t)$$

is not time consistent.

Proof. Consider the first o
which are given in (7) and rep

$$u_1''(c_t)\dot{c}_t + u_1'(c_t)q$$
$$\dot{s}_t =$$

Let $(\widetilde{c}_t, \widetilde{s}_t)_{t=0,\infty}$ be a solution
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be a solution to the problem
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Then the original problem solved at $t = 0$ is time consistent if and only if $(\widehat{c}_t, \widehat{s}_t)_{t=T,\infty} = (c_t^*, s_t^*)_{t=T,\infty}$, i.e., if the original solution restricted to the period $[T, \infty]$ is also a solution to the problem with initial time T and initial stock s_T^*, for any T.

It is shown in [14] that the solutions to dynamic optimization problems are in general time consistent only if the discount factor is exponential. The following result is an illustration of this fact.

PROPOSITION 9.[14] *The solution to the problem of optimal management of a renewable resource with a discount rate falling asymptotically to zero is not time consistent, i.e., the solution to*

$$\max \alpha \int_0^\infty \{u_1(c) + u_2(s)\}\Delta(t)\,dt + (1-\alpha)\lim_{t\to\infty}\{u_1(c) + u_2(s)\},$$

$$0 < \alpha < 1, \text{ s.t. } \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given, } q(t) = -\frac{\dot{\Delta}(t)}{\Delta(t)},\ \lim_{t\to\infty} q(t) = 0.$$

is not time consistent.

Proof. Consider the first order condition for a solution to this problem, which are given in (7) and repeated here with the substitution $\lambda = u_1'$:

$$\left.\begin{array}{c} u_1''(c_t)\dot{c}_t + u_1'(c_t)q(t) = -u_2'(s_t) - u_1'(c_t)r'(s_t) \\ \dot{s}_t = r(s_t) - c_t \end{array}\right\}. \tag{9}$$

Let $(\widetilde{c}_t, \widetilde{s}_t)_{t=0,\infty}$ be a solution computed at date $t = 0$. The rate of change of consumption on this at a date $T > 0$ will be given by (9). Now let $(\overline{c}_t, \overline{s}_t)_{t=\Upsilon,\infty}$ be a solution to the problem with starting date $\Upsilon, 0 < \Upsilon < T$, and initial conditions at Υ given by $(\widetilde{c}_\Upsilon, \widetilde{s}_\Upsilon)$. When the problem is solved again with starting date Υ, the value of $\Delta(t)$ at calendar time Υ is $\Delta(0)$, the initial value of the discount factor. Hence on this path the value of $\Delta(t)$ at date T is $\Delta(T - \Upsilon)$, while it is $\Delta(T)$ on the initial path. Hence $q(T)$ will differ, and the two paths will have different rates of change of consumption for all dates in excess of Υ. This establishes that if the optimum is recomputed at any date $T > 0$, then the initial plan will no longer be followed. □

These are interesting and surprising results: to ensure the existence of an optimal path which balances present and future "correctly" according to Chichilnisky's axioms, we have to accept paths which are not time consistent. Of course, the empirical evidence cited above implies that individual behavior must also be inconsistent, so society in this case is only replicating what individuals apparently do. Traditionally, welfare economists have always regarded time consistency as a very desirable property of intertemporal choice. More recently, this presumption has been questioned: philosophers and psychologists have noted that the same person at different stages of her or his life can reasonably be thought of as different people with different perspectives on life and different experiences.[15] The implications of working with inconsistent choices clearly need further research.

6. Capital and Renewable Resources

Now we consider the most challenging, and perhaps most realistic and rewarding, of all cases: an economy in which a resource which is renewable and so has its own dynamics can be used together with produced capital goods as an input to the production of an output. The output in turn can as usual in growth models be reinvested in capital formation or consumed. The stock of the resource is also a source of utility to the population. So capital accumulation occurs according to

$$\dot{k} = F(k, \sigma) - c$$

and the resource stock evolves according to

$$\dot{s} = r(s) - \sigma,$$

where k is the current capital stock, σ the rate of use of the resource in production, and $F(k, \sigma)$ the production function. As before $r(s)$ is a growth function for the renewable resource, indicating the rate of growth of this when the stock is s.

As before, we shall consider the optimum according to the utilitarian criterion, then characterize the green golden rule, and finally draw on the results of these two cases to characterize optimality according to Chichilnisky's criterion.

7. The Utilitarian Optimum

The utilitarian optimum in this framework is the solution to

$$\left. \begin{array}{l} \max \int_0^\infty u(c_t, s_t) e^{-\delta t} dt \text{ subject to} \\ \dot{k} = F(k, \sigma) - c \text{ and } \dot{s} = r(s) - \sigma \end{array} \right\}. \tag{10}$$

We proceed in the by-now standard manner, constructing the Hamiltonian

$$H = u(c, s) e^{-\delta t} + \lambda e^{-\delta t} \{F(k, \sigma) - c\} + \mu e^{-\delta t} \{r(s) - \sigma\}$$

and deriving the following conditions which are necessary for a solution to (10):

$$u_c = \lambda, \tag{11}$$

$$\lambda F_\sigma = \mu, \tag{12}$$

$$\dot{\lambda} - \delta\lambda = -\lambda F_k, \tag{13}$$

$$\dot{\mu} - \delta\mu = -u_s - \mu r_s, \tag{14}$$

where r_s is the derivative of r with respect to the stock s.

7.1. *Stationary Solutions*

A little algebra shows th
underlying differential e(
solution:

$$\delta = F_k (k$$

$$\sigma = r(s)$$

$$c = F(k$$

$$\frac{u_s(c, s)}{u_c(c, s)} = F_\sigma (k$$

This system of four equat
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It is important to unde
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First, consider this relat
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$$c = F(k, \sigma(s)),$$

and so we have

$$\frac{\partial c}{\partial s}_{k \text{ fixed}} = F_\sigma r_s.$$

As F_σ is always positive,
and then switches to neg
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In general, however, k
on σ via Equation (15). T
as an implicit function, w
across stationary states:

$$\frac{dc}{ds} = r_s \left(-\delta \frac{F_{k\sigma}}{F_{kk}} \right.$$

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stock s.

7.1. *Stationary Solutions*

A little algebra shows that the system (11) to (14), together with the two underlying differential equations in (10), admits the following stationary solution:

$$\delta = F_k(k, \sigma), \tag{15}$$

$$\sigma = r(s), \tag{16}$$

$$c = F(k, \sigma), \tag{17}$$

$$\frac{u_s(c, s)}{u_c(c, s)} = F_\sigma(k, \sigma)(\delta - r_s). \tag{18}$$

This system of four equations suffices to determine the stationary values of the variables k, s, σ and c.

It is important to understand fully the structure of stationary states in this model, and in particular the trade-off between consumption c and the resource stock s across alternative stationary states.

First, consider this relationship across stationary states for a given value of the capital stock k: in this case we can write

$$c = F(k, \sigma(s)),$$

and so we have

$$\frac{\partial c}{\partial s}_{k \text{ fixed}} = F_\sigma r_s. \tag{19}$$

As F_σ is always positive, this has the sign of r_s, which is initially positive and then switches to negative: hence we have a single-peaked relationship between c and s for fixed k across stationary states. The c–s relationship across stationary states for a fixed value of k replicates the shape of the growth function $r(s)$ and so has a maximum for the same value of s.

In general, however, k is not fixed across stationary states, but depends on σ via Equation (15). Taking account of this dependence and treating (15) as an implicit function, we obtain the total derivative of c with respect to s across stationary states:

$$\frac{dc}{ds} = r_s\left(-\delta\frac{F_{k\sigma}}{F_{kk}} + F_\sigma\right), \tag{20}$$

which, maintaining the assumption that $F_{k\sigma} \geq 0$, also has the sign of r_s and again inherits the shape of $r(s)$. Note that for a given value of s: $|(dc/ds)| \geq |(\partial c/\partial s)_{k \text{ fixed}}|$ and that the two are equal only if the cross derivative $F_{k\sigma}$ is zero. The various curves relating c and s across stationary solutions are shown in Figure 5: for $F_{k\sigma} > 0$ the curve corresponding to k fully adjusted to s both rises and falls more sharply than the others, and crosses each of these twice, from below while increasing and from above while decreasing, as shown. A stationary solution with a capital stock $\widehat{k}$ must lie on the intersection of the

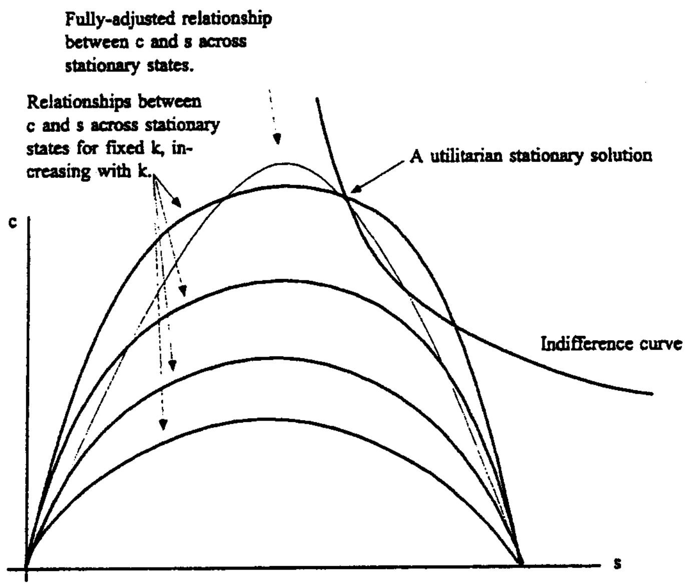

Figure 5. A utilitarian stationary solution occurs where the fully-adjusted c–s relationship crosses the same relationship for the fixed value of k corresponding to the stationary solution.

curve corresponding to a capital stock fixed at $\hat{k}$ with the curve representing the fully-adjusted relationship. At this point, c, s and k are all fully adjusted to each other. (In the case of $F_{k\sigma} = 0$ the curves relating c and s for k fixed and fully adjusted are identical, so that in the case of a separable production function the dynamics are simpler, although qualitatively similar.)

The stationary first order condition (18) relates most closely to the curve connecting c and s for a fixed value of k (the only relevant curve for $F_{k\sigma} = 0$), and would indicate a tangency between this curve and an indifference curve if the discount rate δ were equal to zero. For positive δ, the case we are considering now, the stationary solution lies at the point where the c–s curve for the fixed value of k associated with the stationary solution crosses the c–s curve along which k varies with s. At this point, an indifference curve crosses the fixed-k c–s curve from above: this is shown in Figure 5. Note that as we vary the discount rate δ, the capital stock associated with a stationary solution will alter via Equation (15), so that in particular lowering the discount rate will lead to a stationary solution on a fixed-k c–s curve corresponding to a larger value of k and therefore outside the curve corresponding to the initial lower discount rate.

7.2. Dynamics of the Utilitarian Solution

The four differential equations governing a utilitarian solution are

$$\left.\begin{aligned} \dot{k} &= F(k,\sigma) - c(s_t,\lambda_t) \\ \dot{s} &= r(s) - \sigma(\mu_t,\lambda_t,k_t) \\ \dot{\lambda} - \delta\lambda &= -\lambda F_k \\ \dot{\mu} - \delta\mu &= -u_s - \mu r_s \end{aligned}\right\}.$$

The matrix of the linearized system is

$$\begin{bmatrix} \delta - F_\sigma \lambda \frac{F_{\sigma k}}{F_{\sigma\sigma}} & \frac{u_{cs}}{u_{cc}} & -\frac{F_\sigma F_\sigma}{\lambda F_{\sigma\sigma}} - \frac{1}{u_{cc}} & \frac{F_\sigma}{\lambda F_{\sigma\sigma}} \\ \frac{F_{\sigma k}}{F_{\sigma\sigma}} & r_s & \frac{F_\sigma}{\lambda F_{\sigma\sigma}} & -\frac{1}{\lambda F_{\sigma\sigma}} \\ -\lambda F_{kk} + F_{k\sigma}\lambda \frac{F_{\sigma k}}{F_{\sigma\sigma}} & 0 & \delta + \frac{F_{k\sigma}F_\sigma}{F_{\sigma\sigma}} & -\frac{F_{k\sigma}}{F_{\sigma\sigma}} \\ 0 & \frac{u_{sc}u_{cs}}{u_{cc}} - u_{ss} - \mu r_s & -\frac{u_{sc}}{u_{cc}} & \delta - r_s \end{bmatrix}.$$

To establish clear general results on the signs of the eigenvalues of this matrix, we have to make simplifying assumptions. If $F_{\sigma\sigma}$ is large, so that the marginal productivity of the resource drops rapidly as more of it is employed, then the eigenvalues of the above matrix are : r_s, $\delta - r_s$, $1/(2u_{cc})(2u_{cc}\delta \pm \sqrt{u_{cc}^2\delta^2 + 4u_{cc}\lambda F_{kk}}$. There are two negative roots in this case, as $r_s < 0$ at a stationary solution. In this case the utilitarian stationary solution is locally a saddle point.

PROPOSITION 10.[16] *A sufficient condition for the utilitarian stationary solution to be locally a saddle point is that $F_{\sigma\sigma}$ is large, so that the marginal productivity of the resource diminishes rapidly in production.*

There are other cases in which the stationary solution is locally a saddle point, involving additive separability of the utility function.[17] Existence of a solution to the utilitarian problem is established in the Appendix.

8. The Green Golden Rule with Production and Renewable Resources

Across stationary states, the relationship between consumption and the resource stock satisfies the equation

$$c = F(k, r(s)),$$

so that at the green golden rule we seek to maximize the sustainable utility level with respect to the inputs of capital k and the resource stock s:

$$\max_{s,k} u(F(k, r(s)), s).$$

Maximization with respect to the resource stock gives

$$\frac{u_s}{u_c} = -F_\sigma r_s, \tag{21}$$

which is precisely the condition (18) characterizing the stationary solution to the utilitarian conditions for the case in which the discount rate δ is equal to zero. So, as before, the utilitarian solution with a zero discount rate meets the first order conditions for maximization of sustainable utility with respect to the resource stock. Of course, in general the utilitarian problem may have no solution when the discount rate is zero. Note that the condition (21) is quite intuitive and in keeping with earlier results. It requires that an indifference curve be tangent to the curve relating c to s across stationary states for k fixed at the level $\overline{k}$ defined below: in other words, it again requires equality of marginal rates of transformation and substitution between stocks and flows.

The capital stock k in the maximand here is independent of s. How is the capital stock chosen? In a utilitarian solution the discount rate plays a role in this through the equality of the marginal product of capital with the discount rate (15): at the green golden rule there is no equivalent relationship.

We close the system in the present case by supposing that the production technology ultimately displays satiation with respect to the capital input alone: for each level of the resource input σ there is a level of capital stock at which the marginal product of capital is zero. Precisely,

$$\overline{k}(\sigma) = \min k : \frac{\partial F}{\partial k}(k, \sigma) = 0. \tag{22}$$

We assume that $\overline{k}(\sigma)$ exists for all $\sigma \geq 0$, is finite, continuous and non-decreasing in σ. Essentially assumption (22) says that there is a limit to the extent to which capital can be substituted for resources: as we apply more and more capital to a fixed input of resources output reaches a maximum above which it cannot be increased for that level of resource input. In the case in which the resource is an energy source, this assumption was shown by Berry et al. [5] to be implied by the second law of thermodynamics: this issue is also discussed by Dasgupta and Heal [11]. In general, this seems a very mild and reasonable assumption. Given this assumption, the maximization of stationary utility with respect to the capital stock at a given resource input,

$$\max_k u(F(k, r(s)), s)$$

requires that we pick the capital stock at which satiation occurs at this resource input, i.e., $k = \overline{k}(r(s))$. Note that

$$\frac{\partial \overline{k}(r(s))}{\partial s} = \frac{\partial \overline{k}(\sigma)}{\partial \sigma} r_s;$$

so that $\overline{k}$ is increasing and then decreasing in s across stationary states: the derivative has the sign of r_s. In this case, the green golden rule is the solution to the following problem:

$$\max_s u\left(F\left(\overline{k}(r(s)), r(s)\right), s\right),$$

where at each value of the resource stock s the input and the resource and the capital stock are adjusted so that the resource stock is stationary and

the capital stock maximiz
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$\overline{k}(r(s))$ has the following

$$\frac{dc}{ds} = F_k \frac{\partial \overline{k}}{\partial \sigma} r_s + \ldots$$

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PROPOSITION 11.[18] *In a*
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9. Optimality for the Chi

Now we seek to solve the p

$$\max \alpha \int_0^\infty u(c_t, \ldots$$
$$\text{s.t. } \dot{k} = F(k_t, \sigma_t)$$

the capital stock maximizes output for that stationary resource input. The relationship between consumption and the resource stock across stationary states when at each resource stock the capital stock is adjusted to the level $\overline{k}(r(s))$ has the following slope:

$$\frac{dc}{ds} = F_k \frac{\partial \overline{k}}{\partial \sigma} r_s + F_\sigma r_s$$

and by the definition of $\overline{k}$ the first term on the right is zero, so that the slope of the curve relating c and s when the capital stock is given by $\overline{k}$ is the same as the slope when the capital stock is fixed. This curve is thus the outer envelope of the curves for fixed values of the capital stock.

The total derivative of the utility level with respect to the stock of the resource is now

$$\frac{du}{ds} = u_c F_k \frac{\partial \overline{k}}{\partial \sigma} r_s + u_c F_\sigma r_s + u_s.$$

By assumption (22) and the definition of $\overline{k}$, $F_k = 0$ here: hence equating this to zero for a maximum sustainable utility level gives the earlier expression (21). The green golden rule is characterized by a tangency between an indifference curve and the outer envelope of all curves relating c to s across stationary states for fixed capital stocks.

PROPOSITION 11.[18] *In an economy with capital accumulation and renewable resources, under the assumption (22) of satiation of the production function with respect to capital, the green golden rule satisfies the first order condition* $u_s/u_c = -F_\sigma r_s$ *which defines a tangency between an indifference curve and the outer envelope of c–s curves for fixed values of k. It has a capital stock of* $\overline{k}(\sigma(s^*))$*, where* s^* *is the green golden rule value of the resource stock and* $\overline{k}$ *denotes the capital stock at which the marginal product of capital first becomes zero for a resource input of* $\sigma(s^*)$.

What if production does not display satiation with respect to the capital stock? In this case there is no maximum to the output which can be obtained from a given resource flow and so from a given resource stock. Unless we assume satiation of preferences with respect to consumption, the green golden rule is not well defined.[19]

9. Optimality for the Chichilnisky Criterion

Now we seek to solve the problem

$$\left.\begin{array}{l} \max \alpha \displaystyle\int_0^\infty u(c_t, s_t) e^{-\delta t} dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t) \\ \text{s.t. } \dot{k} = F(k_t, \sigma_t) - c_t \;\&\; \dot{s}_t = r(s) - \sigma_t,\; s_t \geq 0 \quad \forall t. \end{array}\right\}. \tag{23}$$

In the case of satiation of the production process with respect to capital, as captured by assumption (22), the situation resembles that with the Chichilnisky criterion with renewable resources above: there is no solution unless the discount rate declines to zero. Formally:

PROPOSITION 12.[20] *Assume that condition (22) is satisfied. Then problem (23) has no solution.*

Proof. The structure of the proof is the same as that used above. The integral term is maximized by the utilitarian solution, which requires an asymptotic approach to the utilitarian stationary state. The limit term is maximized on any path which asymptotes to the green golden rule. Given any fraction $\beta \in [0, 1]$ we can find a path which attains the fraction β of the payoff to the utilitarian optimum and then approaches the green golden rule. This is true for any value of $\beta < 1$, but not true for $\beta = 1$. Hence any path can be dominated by another corresponding to a higher value of β. □

We now consider instead optimization with respect to Chichilnisky's criterion with a discount rate which declines to zero over time:

PROPOSITION 13.[21] *Consider the problem*

$$\max \alpha \int_0^\infty u_1(c, s)\, \Delta(t)\, dt + (1-\alpha) \lim_{t\to\infty} u_1(c, s), 0 < \alpha < 1,$$
$$s.t.\ \dot{k} = F(k_t, \sigma_t) - c_t \ \&\ \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given.}$$

where $q(t) = -(\dot{\Delta}(t)/\Delta(t))$ *and* $\lim_{t\to\infty} q(t) = 0$. *Any solution to this problem is also a solution to the problem of maximizing* $\int_0^\infty u_1(c, s)\, \Delta(t)\, dt$ *subject to the same constraint. In words, solving the utilitarian problem with the variable discount rate which goes to zero solves the overall problem.*

Proof. The proof is a straightforward adaptation of the proof of Proposition 6, and is omitted. □

As before, the existence of a solution is established in the Appendix.

What does the Chichilnisky-optimal path look like in this case? It is similar in general terms to the set of paths shown in Figure 4, except that the graph of the growth function $r(s)$ is replaced by the outer envelope of the curves relating c and s for fixed values of k. The optimal path moves towards the green golden rule, which is a point of tangency between an indifference curve and the outer envelope of the curves relating c and s for fixed values of k. This point is the limit of utilitarian stationary solutions as the associated discount rate goes to zero.

10. Conclusions

A review of optimal pattern
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11. Appendix

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10. Conclusions

A review of optimal patterns of use of renewable resources has suggested interesting conclusions. The green golden rule is an attractive configuration: it gives the highest sustainable utility level. Utilitarian solutions with a positive discount rate will accumulate a smaller stock of the resource than that associated with the green golden rule, although the difference goes to zero as the discount rate used in the utilitarian formulation gets smaller. Of course, for a zero discount rate, there is typically no utilitarian optimum. Investigation of Chichilnisky's criterion in some measure bridges the gap between these two concepts of optimality: a solution exists if and only if the discount rate in the integral term of Chichilnisky's maximand declines asymptotically to zero, in which case maximization of the integral term alone – the sum of discounted utilities – leads one to the green golden rule. This result remains true even with the inclusion of production: matters are more complex in that case, but not qualitatively different. Interestingly, there is empirical evidence that people display declining discount rates in their behavior towards the future. Such behavior is quite consistent with behavior patterns found in other aspects of human choice and summarized as the Weber–Fechner law.

11. Appendix

In this appendix we establish conditions sufficient for the existence of solutions to the various intertemporal optimization problems considered in Propositions 2, 6, 7, 10 and 13 of the text. We use an approach and a set of results developed initially by Chichilnisky [6] and applied by Chichilnisky and Gruenwald [9]. This is a very direct and intuitive approach: we show that the set of feasible solutions to the constraints is a compact set, and that the objective function is a continuous function, and invoke the standard result that a continuous function on a compact set attains a maximum. The delicate step here is to find a topology in which we have compactness and continuity under reasonable assumptions about the problem: for this we use weighted L_p spaces, as introduced in Chichilnisky [6].

We consider the utilitarian optimality problem analyzed in Section 7, as this is the most complex of the problem in the paper. Earlier problems in the paper are special cases of this, so that the existence of a solution to this implies the existence of solutions to the earlier problems. The optimization problem is:

$$\left.\begin{array}{l}\max \displaystyle\int_0^\infty u\left(c_t, s_t\right) e^{-\delta t} dt \text{ subject to} \\ \dot{k} = F\left(k, \sigma\right) - c \text{ and } \dot{s} = r\left(s\right) - \sigma\end{array}\right\}. \tag{24}$$

We make the following *assumptions*:

1. $u(c,s)$ is concave, increasing and differentiable. It satisfies the Caratheodory condition, namely it is continuous with respect to c and s for almost all t and measurable with respect to t for all values of c and s.
2. $r(0) = 0$, $\exists \overline{s} > 0$ $s.t.$ $r(s) = 0 \, \forall s \geq \overline{s}$, $\max_s r(s) \leq b_1 < \infty$, and $r(s)$ is concave for $s \in [0, \overline{s}]$.
3. For any $\sigma \exists b_2(\sigma) < \infty$ $s.t.$ $F(k,\sigma) \leq b_2(\sigma)$.
4. $\exists b_3 < \infty$ $s.t.$ $|\dot{s}| \leq b_3$.
5. $\exists b_4 < \infty$ $s.t.$ $\left|\dot{k}\right| \leq b_4$.

The first two conditions are conventional. The third implies that bounded resource availability implies bounded output: it is a form of the assumption made by Dasgupta and Heal [10] that the resource is essential to production. It is a restatement of assumption (22) in the text. The final two assumptions imply that it is not possible for either the resource stock or the capital stock to change infinitely rapidly. These seem to be very reasonable assumptions. However, we shall in the end not require them: we shall prove the existence of an optimal path under these assumptions, and then note that a path which is optimal without these assumptions is still feasible and optimal with them.

PROPOSITION 14. *Under assumptions (1) to (5) above, the utilitarian optimization problem (24) has a solution.*

Proof. Under the above assumptions, the set of feasible time paths of the resource stock s and consumption c are uniformly bounded above. (Note that s is bounded by (2), and c by (3) and (5).) They are non-negative and so bounded below. Hence the paths of s and c are integrable against some finite measure and so are elements of a weighted L_1 space. Denote by P the set of feasible paths s_t and c_t, $0 \leq t \leq \infty$: as a subset of L_1, P is closed and norm bounded, so that by the Banach–Alaoglu theorem it is weak-* compact. By Lebesgue's bounded convergence theorem, it is also compact in the norm of L_1.

The objective $U = \int_0^\infty u(c_t, s_t) e^{-\delta t} dt$ maps P to the real line $\Re$. To complete the proof we need to show that U is continuous in the norm of L_1. This follows immediately from the characterization of L_p continuity given in [6]:

LEMMA 15 (Chichilnisky). *Let $W = \int_\Re u(c_t, t)\, d\nu(t)$ for a finite measure $\nu(t)$, with $u(c_t, t)$ satisfying the Caratheodory condition. Then W defines a norm-continuous function from L_p to $\Re$ for some coordinate system of L_p if and only if $|u(c_t, t)| \leq a(t) + b|c_t|^p$, where $a(t) \geq 0$, $\int_\Re a(t)\, d\nu(t) < \infty$ and $b > 0$.*

In the case of our objective the role of $u(c_t, t)$ is played by $u(c_t, s_t) e^{-\delta t}$. An extension of Chichilnisky's lemma to functions u defined on $\Re^2$ is straightforward. As u is defined only on $\Re^2_+$, concavity implies that Chichilnisky's

inequality is satisfied for p
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We have now proven the ex
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1. See [12] for a detailed listing
2. This proposition, which was
3. Elsewhere we have called thi
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inequality is satisfied for $p = 1$. This completes the proof of existence of an optimum. □

We have now proven the existence of an optimal path for the most complex of the optimization problems discussed in the paper: existence of an optimum for the simpler problems can be deduced from this. Our proof used assumptions (4) and (5) above, which bound respectively $\dot{s}$ and $\dot{k}$, the rates of change of the resource and capital stocks. These assumptions were not made in the body of the paper. However, note from the characterization results in the paper that solutions to the problems without bounds on the rates of change of stocks do in fact have bounded rates of change of the stocks of the resource and of capital. Hence for sufficiently large bounds, the imposition of bounds on the rates of change of stocks cannot change the solutions to the optimization problems. It follows that we have also established the existence of solutions for the unbounded optimization problems.

Notes

1. See [12] for a detailed listing of many more examples.
2. This proposition, which was first proved in [18], is a strengthening of results in [3].
3. Elsewhere we have called this the green golden rule [3].
4. This result, and the associated concept of the green golden rule, were introduced in [1] and [3].
5. We are grateful to Kenn Judd for this observation.
6. This result was introduced in [18].
7. We are grateful to Harl Ryder for suggesting this result and outlining the intuition behind it.
8. This result was first proven in [18].
9. This equality is not always true: it requires locally uniform convergence of the non-autonomous system to the autonomous system. For details, see [4].
10. This result was first proven in [18].
11. Due to Harl Ryder.
12. This discount factor is infinite when $t = 0$: hence the need to start from $t = 1$.
13. Further discussions of time consistency can be found in [14].
14. This result was first proven in [18].
15. For a further discussion, see [13] and references therein.
16. This result was first proven in [18].
17. See [18] for details.
18. This result was first proven in [18].
19. See [18] for details.
20. This result was first proven in [18].
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RALPH ABRAHAM, GRAC

2.2. North-South Tra
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1. Introduction

This paper develops a dyr
environment plays an impc
North-South model for the
of the world economy. Th
We introduce dynamics in
endogenous accumulation
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the approach to equilibrium
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evolution of capital stock th

G. Chichilnisky et al. (eds.), Sustainat