JOURNAL OF
Mathematical
ECONOMICS

ELSEVIER Journal of Mathematical Economics 29 (1998) 27-42

Smooth infinite economies

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Accepted 1 January 1997

Abstract

Equilibrium conditions in smooth infinite economies with separable utilities are described by Fredholm maps, which are Frechet differentiable. Therefore, Smale's extension of Sard's theorem can be used to study infinite economies. We study structural stability and local uniqueness of equilibrium in smooth infinite economies and relate the theory of markets to modern Fredholm theory. C 1998 Elsevier Science S.A.

JEL classification: D52

Keywords: General equilibrium; Fredholm map; Infinite economy

1. Introduction

Many problems in economics involve infinite dimensions. Typical examples are dynamic choices, such as optimal portfolios in finance and optimal paths in growth models (see, for example, Chichilnisky and Kalman, 1980, Duffie and Huang, 1985). In the past 2 decades there has been an increasing interest in the study of infinite economies. However, much of the literature so far has mainly dealt with existence of equilibrium; determinacy of equilibrium is largely unexplored. Chichilnisky and Kalman (1980) have dealt with a special case in the context of resource allocation problems. Kehoe et al. (1989) have followed an approach that takes excess demand functions as primitives. This paper proposes to attack the

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PII S0304-4068(97)00009-8

problem by taking preferences and endowments as primitives. We set up a framework for analysis and relate determinacy theory to modern Fredholm theory. This is useful because differential topology can be extended to infinite dimensional manifolds with Fredholm maps. Under these conditions a locally stable equilibrium corresponds to a point in the domain of some Fredholm map, at which the Fredholm index is equal to zero. The central issues in this approach are the following:

1. To identify a `right' framework, which is both mathematically tractable and economically meaningful. In other words, to choose a `right' topological space, where Fredholm theory can be applied and which is natural from an economic point of view. These two requirements are usually not satisfied simultaneously. For example, the positive cone of Hilbert space L^2_+ is natural in financial markets but not good for Fredholm theory because it has an empty interior. [1]
2. To characterize conditions for the parameters of dynamic economic systems such that the underlying economies are `well behaved'. To apply Fredholm theory, the excess demand function of the economy must be a Fredholm map between two infinite dimensional manifolds. This property cannot be `assumed': it must be derived from the parameters of the economy. Even continuity is not guaranteed. Excess demand functions in infinite economies are typically not well defined and are generally not smooth even when they are well defined. Indeed, following Chichilnisky (1976), and subsequently Araujo (1987), excess demand functions could be smooth only if the commodity space is a Hilbert space, whose natural positive cone usually has an empty interior.

In order to use differential techniques, we assume that the commodity space L is a separable topological vector space for which the interior intL_+ of the positive cone L_+ is non-empty. Typically examples of such spaces are C([0, T]) with the supremum norm $\|x\| = \sup_{t \in [0,T]} |x(t)|$ and the positive cone $C_+([0, T]) = \{x: x(t) \geq 0, \forall t \in [0, T]\}$. For simplicity, in this paper we restrict ourselves to the commodity space C(M, R^n), where M is any compact manifold. For more general treatment of commodity spaces, see Chichilnisky and Zhou (1995).

The literature typically takes as the price space the natural positive cone L'_+ of the dual space L' of L. When the positive cone of the underlying commodity space has a non-empty interior, the corresponding price space is extremely large, which is the reason that excess demand functions cannot be well defined. However, not all elements of the price space are equally interesting. With separable utility functions, only a small subset of the price space can support equilibria. Therefore, there is no loss of information from discarding those

[1] See Chichilnisky (1976, 1977) and Chichilnisky and Heal (1993) for the introduction of Hilbert space and Sobolev space (such as L^2) in economic models. In Fredholm theory, the domain and range of Fredholm maps must have the structure of Banach manifolds. Since L_+ has an empty interior, it is not a Banach manifold, therefore, L^2 is not an ideal space for Fredholm theory because prices are always in L^2_+.

elements that do not support equilibria. This is the key step in our departure from the traditional way of thinking about this problem, which makes it possible to bring differential techniques to infinite spaces.

We show that, for separable utility functions on infinite dimensional Banach spaces, equilibrium conditions are described by Fredholm maps, which are Frechet differentiable. Therefore, Smale's extension of Sard's theorem can be used to obtain the determinacy theory of equilibrium. [2]

The next section will set up a general framework that we will work with. Sections 3-5 report the main results in the paper. The last section makes some concluding remarks and lists our plans for further research.

2. Preliminaries and mathematical notation

In this section we group together some basic mathematical definitions, notations and facts that will be used later.

2.1. Fredholm index theory

The most important mathematical concept that we will use is the Fredholm map. Here we only give a brief review. Readers may refer to Yosida (1974) or Conway (1985) for details.

Given two Banach spaces X and Y, the vector space of all bounded linear maps from X to Y will be denoted by $L(X, Y)$, with the norm $\|\cdot\|$ defined by

$$\|T\| = \sup\{\|Tx\| : \|x\| \le 1\}$$

L(X, Y) is a Banach space. Let M be a closed subspace of X. Define

$$\operatorname{Codim} M = \operatorname{Dim}(X/M)$$

Let $T \in L(X, Y)$, and

$$\operatorname{Ker} T = \{x \in X, Tx = 0\}$$

$$\operatorname{Ran} T = \{y \in Y, Tx = y, x \in X\}$$

The map T is said to be a Fredholm operator if, and only if, RanT is closed and

$$\operatorname{Dim}(\operatorname{Ker} T) < \infty, \quad \operatorname{Codim}(\operatorname{Ran} T) < \infty$$

The index of T is defined by

$$\operatorname{Ind} T = \operatorname{Dim}(\operatorname{Ker} T) - \operatorname{Codim}(\operatorname{Ran} T)$$

We denote by $D(X, Y) \in L(X, Y)$ all Fredholm operators. A linear map $T \in L(X, Y)$ is compact if $T(x, \|x\| < 1)$ has compact closure in Y. The set of compact

[2] In addition to the papers we have mentioned, there are other papers dealing with determinacy theory of equilibrium in infinite dimension, using different approaches. See, for example, Dana (1994) and Shannon (1994).

operators from X into Y is denoted by $L_0(X, Y)$. $T \in L(X, Y)$ has finite rank if RanT is finitely dimensional. Clearly any linear map that has finite rank is compact. An interesting fact about Fredholm operators is that their index is invariant under compact perturbations.

Theorem 1. *Let* $A \in D(X, Y)$, *and suppose that* $K \in L_0(X, Y)$. *Then* $A + K \in D(X, Y)$ *and* $\mathrm{Ind}(A + K) = \mathrm{Ind}\, A$.

Theorem 2. *Let X, Y, Z be Banach spaces and suppose that* $A \in D(X, Y)$ *and* $B \in D(Y, Z)$, *then* $AB \in D(X, Z)$ *and*

$$\mathrm{Ind}(AB) = \mathrm{Ind}\, A + \mathrm{Ind}\, B$$

Theorem 3. *If* A is *invertible and* $K \in L_0(X, Y)$, *then* $\mathrm{Ind}(A + K) \sim 0$.

2.2. Basic non-linear analysis

We assume the readers are familiar with basic calculus on Banach space. A basic reference is Zeidler (1985), or any standard functional analysis textbooks.

First we note that there are similar implicit mapping and inverse mapping theorems on Banach spaces, see Abraham and Robbin (1967)

Theorem 4 (implicit mapping theorem). *Let X, Y, Z be Banach spaces, U is open in X, V is open in Y,* $f: U \times V \to Z$ *be a* C^m*-map* $(m > 0)$. *Let* $f_2'(x_0, y_0): Y \to Z$ *be a homeomorphism for* $(x, y) \in U \times V$, *where* $f_2'(x_0, y_0)$ *represents the partial derivative of f with respect to the second variable y. Then there is an open neighborhood* U_0 *of* x_0 *and* C^m*-map* $g: U_0 \to V$ *such that* $g(x_0) = y_0$ *and* $f(x, g(x)) = f(x_0, y_0)$ *for all* $x \in U_0$. *g is unique.*

Theorem 5 (inverse mapping theorem). *Let X, Y be Banach spaces, and U be an open set in X.* Assume *that* $f: U \to Y$ *be a* C^m-map$(m > 0)$. If $f'(x_0)$ is a *homeomorphism* at $x_0 \in U$, *then f is* a local homeomorphism at x_0.

Let X and Y be Banach manifolds and $f: X \to Y$ be a C^1-map. The Frechet derivative of f is denoted by f', and the `differential' of f is denoted by Df [3]. We shall say that f is a Fredholm map if for all $x \in X$, the linear map $f'(x): T_x X \to T_{f(x)} Y$ is a Fredholm operator. If X is connected, the index of $f'(x)$ will not depend on the particular choice of the point x in X and is referred to the

[3] For a definition of the Frechet derivative and the differential of a map on Banach manifolds, see Abraham and Robbin (1967) or Zeidler (1985).

index of f. The following Sard-Smale's theorem will be the main tool to explore the generic property of local uniqueness of competitive equilibrium.

Theorem 6 (Sard-Smale's theorem). Let $f: X \to Y$ be a C^m ($m > 0$) Fredholm map between separable Banach manifolds X and Y. Assume Y is complete, X connected, and $m >$ index(f). Then the set of regular values for f contains a dense G_δ subset of Y. A G_δ subset is defined as the intersection of a countable family of open subsets.

3. The market

The model represents a pure exchange economy with infinitely many commodities and a finite number of consumers. It is derived from economic considerations and can be summarized as follows:

The economy is denoted

$$\_ (W i'170X i)i-<t\_t$$

There are I agents, index by i. X_i is the consumption set of agent i. For simplicity, in this paper we may assume that $X_i = C_{++}(M, R^n)$. $W(x)$ is the utility function of agent i. ϖ_i is the initial endowment of agent i. Society's endowment ru=Yivi $\in C_{++}(M, R^n)$. We assume that the utility functions are separable, i.e. they can be written as the following form

$$W(x) = \int_M u^i(x(t),t)\,dt$$

where the integral is with respect to some metric on the compact manifold M.[4]

Example 1. In growth models, the utility function $W(x)$ is simply a continuous-time version of a discounted sum of time-dependent utilities. In finance, when the underlying parameters follow a diffusion process, Wi(x) is just the expectation of state-dependent utilities, and M is the state space.

Since we are going to use differential techniques, we need to find conditions that guarantee that $W(x)$ is twice Frechet differentiable. Let there be given C^2 functions

$$u^i(x,t): R_{++} \times M \to R$$

Given t, let

$$V_t^i(x) = \{y \in R^n_{++} \mid u^i(y,t) \geq u^i(x,t)\}$$

[4] The assumption of compactness is only for the simplicity of exposition, it is by no means necessary. For a general treatment, see Chichilnisky and Zhou (1995).

Assumption 1. For any fixed t $\in$ M, the function $u^i(x, \ ): R^n_+ \to R$ is strictly monotonic and concave, and V,'(x) is closed.

Proposition 1. Under Assumption 1, the fitnction

$$W_i(x) = \int_M u^i(x(t),t)dt$$

is strictly monotonic, concave over X_i, and is twice Frechet differentiable.

Proof. The proof has several steps:

Step 1: We observe that

$$W(x) = \int_M u^i(x(t),t)dt$$

is well defined. It is a simple consequence of Assumption 1.

Step 2: We show that $W_i(x)$ is a C' functional. Take any x(t) $\in C_{++}(M, R^n)$ and write the Gateaux derivative with respect to c(t) $\in C_{++}(M, R^n)$.

$$\frac{d}{da} \text{ as } \int_M u^i(x(t)+av(t),t)dt$$

We can use the Lebesgue theorem on the differentiation of integrals with respect to a parameter

$$\frac{\partial}{\partial a}\int_M u^i(x(t)+at'(t),t)dt = \int_M \frac{\partial}{\partial a} u^i(x(t)+au(t),t)dt$$

Therefore

$$\frac{d}{da} = \int_M u^i_x(x(t),t)\, v(t)dt$$

$$= \langle u^i_x(x(t),t),v(t)\rangle$$

So under the inner product s (- , -) on $C(M, R^n)$, the Frechet derivative of $W_i(x)$, denoted by $W_i'(.)$, can be uniquely expressed as the following form:

$$W_i'(x(t)) = u^i_x(x(t),t)$$

Step 3: To show that $W_i'(x(t))$ is C^1 Frechet differentiable, we recall the definition of Frechet derivative.

We observe that it is not complete.

Definition 1. Given a map 7r: X --> Y, a point x ∈ X and a map A E L(X, Y), we say that A is the Frechet derivative of -rr at x if

$$\lim_{y\to 0}\frac{1}{\|y\|}\|\pi(x+y)-\pi(x)-Ay\|=0$$

where X and Y are Banach spaces.

Now we return to the proof of Step 3.
First, we show that

$$W_i'(x(t)):\ x(t)\to u_x'(x(t),t)$$

is continuous.

Let x„(t) --4 x(t) in $C(M, R'')$, we need to show that

$$\text{it 'V(x}_n\text{(t),t)}\to u_x'(x(t),t)$$

is in $C(M, R'')$. Since |x(t)I < b for some constant b, $u',,(x(t),t)$ is also bounded by the compactness of M. Therefore,

$$\|u'(x_n(t),t)-u'(x(t),t)\|\le b\|x_n(t)-x(t)\|\to 0$$

Now we show that W,'(x(t)) is Frechet differentiable and that

$$W_i''(x(t))=u_{xx}'(x(t),t)$$

Following the definition, we need to show that

$$\lim_{\to 0}\frac{1}{\|y\|}\|u_x'(x(t)\ +Y(t)'t)\ -u\_x(t),t)\ -u_x^{i}(x(t),t)\qquad t)11 =0$$

We note that u',. j x(t), t) is a matrix-valued function of t and that

$$u_x'(x(t)+y(t),t)-u_x'(x(t),t)=It''(x(t)+e(t),t)-y(t)$$

where 0 < 9(t) < v(t). So the above formula can be written as

$$\lim_{y\to 0}\frac{1}{\|y}\ \|[u_{xx}^i(x(t)+\theta(t),t)-u_{xx}^i(x(t),t)]\circ y(t)\|$$

$$<\lim_{0}\|\,u`V.,(x(t)+9(t),t)-u'v.'-(x(t)'t)]\|=0$$

since 9(t) - 0 when $1(t)\to 0$ in $\|\cdot\|$-norm.

4. Excess demand functions

An excess demand function is a map from the price space to the commodity space describing the difference between what is available and what is demanded by all traders at each price. The demand function describes what traders desire and can afford to buy at those prices. In finite L-dimensional spaces, excess demand functions are formally defined as follows:

Definition 2. A function f_i: $R_{++} \to R^L$ is called the excess demand function of agent i if it is defined by letting $x_i = f_i(p) + \varpi_i$ be the unique maximizer [6] of $W_i(x_i)$ subject to the budget constraint $px_i \leq p\varpi_i$.

In the case of infinite dimensional spaces, the existence of excess demand functions can be problematic. The literature typically takes as the price space the natural positive cone of the dual space of the underlying commodity, space. When the positive cone of the underlying commodity space has a non-empty interior, the correspondent price space is extremely large. In this case excess demand functions in infinite economies typically are not well defined.

However, not all elements of the price space are equally interesting. With separable utility functions, only a small subset of the price space can support equilibria. There is no loss of information. from discarding those elements that do not support equilibria. This is the key step in our departure from the traditional way of thinking about this problem, which makes it possible to bring the differential techniques to infinite spaces.

In this section; we will develop this approach. Unlike the current literature, we will restrict our attention to a subset of the price space to define an excess demand function without loss of information. To simplify the exposition, we make the following assumptions:

Assumption 2. When $n = 1$,

$$\lim_{x \to 0^+} u'_x(x, t) = +\infty$$

for each fixed t.

Remark 1. Assumption 2 is widely used in financial literature to obtain interior equilibria. It assumes that instantaneous utility functions have infinite marginal utility for consumption at zero.

When $n > 1$, we would like to make a similar assumption for utility functions. For each fixed t, consider the corresponding indirect utility functions. Denote by $V(p, w, t) = u(x(p, w), t)$ the indirect utility functions, where

$$x(p,w) = \arg\max_{px \leq w} u(x,t), \quad x \in R^n, p \in R_{++}$$

For each fixed p and t, we know $(\partial V)/(\partial w) > 0$.

[6] The uniqueness is not required in the usual definition of excess demand function. It can be guaranteed by the assumption of strict concavity of utility functions.

Assumption 3. When n > 1,

$$\lim_{\omega \to 0} + \frac{av}{aw} = + 00$$

for each fixed t and p.

Proposition 2. Let

$$\mathrm{GradWi}(x(t)) = \frac{ui(x(t),t)}{\| u^i(x(t),t) \|}$$

Under Assumptions 1-3,

$$\mathrm{Grad}W_i(x(t)) : C^{++}(M, R^n) \to S_{++}$$

is C^1 map and onto, where S++ = {y E C++(M, R^n)| ||y|| = 1)

Proof. Let p(t) E S++, and choose some A > 0 such that

$$\min_{t \in M} \mathrm{Ap}(t)$$

is sufficiently large. Let

$$\mathrm{ux}(x(t), t) = \text{'kp}(t)$$

By our assumptions., there is a well-defined x(t) such that the above formula holds. Now we need to show that such x(t) is contained in C ++(M, R^n). To see this, we consider the inverse of UX(x(t), t) with respect to t,

$$x(t) = (u_x^i)^{-1}(\lambda p(t), t)$$

By the implicit mapping theorem, we know that $(u_x^i)^{-1}(c,$ t) is continuous with respect to C and t. The same procedure as in the proof of Proposition 1 completes the proof. □

Definition 3. A price P: C(M, R^n) → R is a real valued linear function on C(M, R n) which gives non-negative value to any element in C ++(M, R°). A feasible allocation $\vec{x}(t) = (x_1(t), \ldots, x_l(t))$ is an allocation such that $\Sigma_{i=1}^{l} x_i < \varpi$. A feasible allocation is an equilibrium when there is a non-zero price P with $P(\varpi_i) = P(x_i)$ and for any y E C++(M, R n),

$$\mathrm{Wi}(y) > \mathrm{Wi}(xi) \,\text{-}), \quad P(y) > P(xi)$$

Existence of equilibrium can be derived from Bewley (1972). ' It is shown in Bewley (1972) that, to obtain equilibrium prices in L,(M, Σ, μ) (a subset of the dual space of L.(M, .X, p,)), s utility functions must be Mackey continuous. [9] He

[7] For a more general discussion of existence of equilibrium in Hilbert space and Sobolev space, see Chichilnisky and Heal (1993).

[8] $L_\infty(M, \Sigma, \mu)$ consists of all essentially bounded functions from M to R".

[9] The Mackey topology is the strongest topology on L, for which L, is the dual of L..

showed that this condition, in addition to the basic assumptions, is indeed sufficient.

Theorem 7 (Bewley). Assume, in addition to Assumptions 1-3, that

(i) *each utility function is Mackey continuous;*

(ii) $\varpi \in$ int L:(M,

Then the economy has a quasi-equilibrium, and every quasi-equilibrium price belongs to $L_{,}(M, .I, \mu)$.

It is shown in Bewley (1972) that separable utility functions considered in this paper are indeed Mackey continuous. Furthermore, from the discussions in previous sections and under Assumptions 1-3, it is easy to show that every quasi-equilibrium is an equilibrium and every equilibrium price belongs to int $L_{\infty}^{+}(M, \Sigma, \mu)$. In the following we further refine Bewley's result. We note that our consumption space $C_{++}(M, R'')$ can be embedded into L: (M, $Y, p,)$. In our case, the state space M is a compact manifold, the σ-algebra Σ is generated by the open sets of M, and the probability measure μ is the given metric on M (of course, we can normalize A, such that $\mu(M) = 1$).

Theorem 8 (refinement of Bewley). If 2t71 $\in C_{++}(M, R'')$, $\forall i$, *then, under Assumptions 1-3, every equilibrium price* $p \in C_{++}(M, R'')$ *and every equilibrium consumption* c_i *of agent i belongs to* $C_{++}(M, R'')$.

Proof. Let $u_i(c_i, t) = \lambda_i$ P, $\forall i$. It is easy to see that A,=* 0. A competitive equilibrium is Pareto efficient by the first welfare theorem, therefore it maximizes a weighted sum of individual utilities. For separable preferences, the weight is just the inverse of the Ais at the optimal consumption. [10] Let

$$u(y,t) = \max_{x_i \in L_{\infty}^{+}} \sum_{i=1}^{I} \frac{1}{\lambda_i} u_i(x_i,t)$$

subject to

It is easily verified that $\bar{u}(y, t)$ satisfies all assumptions about utility functions in previous sections. Let

$$U(Y) = \int_M u(y(t),t)dt$$

[10] See Chi-Fu Huang (1987).

We note that

for some Jt > 0. Therefore, $\varpi \in C^{++}(M, R^n)$ implies that $p \in C^{++}(M, R^n)$.

The next result, combined with Proposition 2, tells us that we can establish similarly individual demand theory in infinite economies for the model specified in this paper.

Proposition 3. For all $p(t) \in S_{++}$, and $w \in R_{++}$, under Assumptions 1-3, there is a unique demand vector $F_i(p(t), w) \in C^{++}(M, R^n)$ such that

$$F_i(p(t),w) = \arg\max_{\langle p(t),y\rangle = w} W_i(y)$$

If and only if

$$u_x^i(F_i(p(t),w),t) = \lambda p(t)$$

$$\langle p(t),F_i(p(t),w)\rangle = w$$

for some $\lambda > 0$, and furthermore,

$$F_i(p(t),t): S_{++} \times R_{++} \to C^{++}(M,R^n)$$

is a diffeomorphism.

Proof. Step 1: Necessity.

It is an immediate consequence of Proposition 2.

Step 2: Sufficiency.

Denote the hyperplane by

$$A = \{y \in C^{++}(M,R^n) | \langle p(t),y\rangle = w\}$$

Obviously, $W_i(y)$ is a strictly concave function on A. *Also,*

$$\left.\frac{\mathrm{d}W_i(F_i + a(y - F_i))}{\mathrm{d}a}\right|_A = \int_M u_x^i(F_i,t)(y - F_i)\mathrm{d}t$$

$$= \lambda \int_M p(t)(y - F_i)\mathrm{d}t$$

$$= 0$$

Therefore,

$$F_i(p,w) = \arg\max_{y \in A} W_i(y)$$

It is unique globally on A by the strict concavity of $W_i(y)$.

Step 3: Denote G_i: $C^{++}(M, R^n) \to S_{++} \times R_{++}$ by

$$G_i(x(t)) = [\mathrm{Grad}W_i(x(t)), \langle \mathrm{Grad}W_i(x(t)), x(t)\rangle]$$

It is easy to check that Gi is differentiable and onto. From Step 2, we know that the composite map

$$F_i \circ G_i: C^{++}(M,R^n) \to C^{++}(M,R^n)$$

is identity map, and

$$G_i \circ F_i: S_{++} \times R_{++} \to S_{++} \times R_{++}$$

is also identity map. Therefore, $F_i^{-1} = G_i$.

Proposition 4. Under Assumptions 1-3, the demand function is a Fredhohn map and the index of F'(p, w) at (p, w) is 0. Furthermore, the index of $F_i'(p, w)$ can be written as the sum of a inverse operator and a finite rank operator.

Proof. $\operatorname{Ind} F_i = 0$ is a direct result of Proposition 3. Now let's calculate F_i' by means of the formula in Proposition 3. Denote by DF_i the differential of F_i. Differentiate the two sides of the following two equations

$$u_x^i(F_i,t) = \lambda p(t)$$
$$\langle p(t),F_i\rangle = w$$

we get

$$u_{xx}^i(F_i,t)\,DF_i = \lambda Dp(t) + D\lambda p(t)$$
$$\langle Dp(t),F_i\rangle + \langle p(t),DF_i\rangle = Dw$$

simplifying the above formula, we get

$$DF_i = \lambda\left(u_{xx}^i\right)^{-1} Dp(t) + D\lambda\left(u_{xx}^i\right)^{-1} p(t) \tag{2}$$

Where $(u_{xx}^i)^{-1}$ is the inverse of matrix $u_{xx}^i(F_i,t)$ for each t. Put Eq. (2) into Eq. (1), we obtain

$$D\lambda\langle p(t),(u_{xx}^i)^{-1}p(t)\rangle = -\lambda\langle p(t),(u_{xx}^i)^{-1}Dp(t)\rangle - \langle Dp(t),F_i\rangle + Dw$$

Therefore,

$$D\lambda = -\frac{\lambda\langle p(t),\left(u_{xx}^i\right)^{-1} Dp(t)\rangle + \langle Dp(t),F_i\rangle - Dw}{\langle p(t),(u_{xx}^i)^{-1} p(t)\rangle} \tag{3}$$

Therefore, substituting $D\lambda$ in Eq. (2) by Eq. (3), we obtain

$$DF_i(p(t),w) = -\frac{\lambda\langle p(t),\left(u_{xx}^i\right)^{-1} Dp(t)\rangle + \langle Dp(t),F_i\rangle}{\langle p(t),\left(u_{xx}^i\right)^{-1} p(t)\rangle}\left(u_{xx}^i\right)^{-1} p(t)$$
$$+ \frac{\left(u_{xx}^i\right)^{-1} p(t)}{\langle p(t),\left(u_{xx}^i\right)^{-1} p(t)\rangle} Dw + \lambda\left(u_{xx}^i\right)^{-1} Dp(t)$$

From Eq. (4) the operator defined by the first term has a finite rank, and the sum of the last two terms defines an invertible operator, therefore, F_i is, by definition, a Fredholm operator.

5. Stability of equilibrium

Now we have all the required mathematical tools in our hand to deal with the main issue in this paper. Since for each agent i, there is a well-defined individual demand function F(p(t), w), we can: define our economy by $S = (F_i, X_i, \varpi_i)_{1 < i \leq I}$. For the rest of the section, we will denote by H = C ++ (M, R"). We assume that the demand functions are fixed, so the economy is actually defined by

$$\varpi = (\varpi_1, \dots, \varpi_I) \in H \times \dots H = H^I$$

Given $\varpi \in H$, if an element $p(t) \in H$ satisfying

$$\sum_{i=I}^{I} F_i(p(t), \langle p(t), \varpi_i \rangle) = \sum_{i=1}^{I} \varpi_i$$

then $p(t)$ is an equilibrium price of the economy. We denote by $E(\varpi)$ the set of $p(t)$ satisfying the above equality.

Theorem 9 (the main theorem). There is a dense G, subset V of H^I, the space of endowments, such that $E(\varpi)$ is discrete for any $\varpi \in V$, and for each such discrete point w, locally the equilibrium in E(v7) depends continuously on w.

Proof. An outline of the proof is as follows. We show that the function F as defined below is a Fredholm map, therefore, we can use the Sard-Smale theorem and follow directly the proof for the finite dimensional case in Debreu (1970). Let

$$\Delta = S_{++} \times R_{++} \times H^{I-1}$$

We define the function F from Δ to H^I by associating with an element $e = (p(t), w, nT_2, \dots, \varpi_I)$ of d the value $F(e) = (z7_1, \dots, \varpi_I)$, where

$$\varpi_1 = F(p(t), w) + \sum_{i=2}^{I} F_i(p(t), \langle p(t), \varpi_i \rangle) - \sum_{i=2}^{I} \varpi_i$$

It is a simple exercise to find that for every $e \in \Delta$, one has $\langle p(t), \varpi_1 \rangle = w$. We also note that, given $\varpi \in H^I$, the equilibrium price $p(t)$ belongs to $E(\varpi)$ if and only if

$$F(p(t), \langle p(t), \varpi_1 \rangle, \varpi_2(t), \varpi_3(t), \dots, \varpi_I(t)) = zg(t)$$

and that the points of $E(\varpi)$ are in one-to-one correspondence with the points of $F^{-1}(\varpi)$. From the last section, we know that F is differentiable. In order to use Smale's theorem, we need to prove that the index of F' is 0.

To see this, let zr_ I = $(zU_2, \dots, \varpi_I)$. Therefore,

$F: (p(t), w, Vs\_1) - (W1, W\_1)$

So we have

F: (T,(t),R,m-I)-(MI,ru_i)

Let

∂F

be the partial derivative with respect to (p(t), w), while ϖ_{-1} remains fixed. In the same way, we define the partial derivative $(\partial F)/(\partial\varpi_{-1})$ with respect to ϖ_{-1}. We directly calculate the partial differential of D(),(t).,,.) F and $D_{\varpi_{-1}}F$ as follows

$$D_{(p(t),w)}F = DF_1 + \sum_{i=2}^{I}\left[\frac{\partial F_i}{\partial p}Dp(t) + \frac{\partial F_i}{\partial w}\langle Dp(t),\varpi_i\rangle\right]$$

From Eq. (4) in the last section, we have

$$D_{(p(t),w)}F = \quad \lambda_i(u^i_{xx}) \quad Dp(t)$$

$$t \quad -\lambda_i\langle p(t),(u^i_{xx})^{-1}Dp(t)\rangle = \langle Dp(t),F_i\rangle + \langle Dp(t),\varpi_i\rangle \quad -1$$

$$Dw - \langle Dp(t),\varpi_1\rangle \quad -1$$

$$\langle p(t),(u^1_{xx}) \quad p(t)\rangle \tag{5}$$

Since for each fixed t, $(u'_{..})^{-'}$ is a negative definite symmetric matrix for each fixed t, hence

t

is an invertible linear operator. As for D_{ϖ} F,

∂F_i

aw

Therefore, we have

$DF(p(t),w,\omega_{-1})$

$$-\lambda_i\langle p(t),(u^i_{xx})^{-1}Dp(t)\rangle - \langle Dp(t),F_i\rangle$$

$$\frac{WDP(t),\varpi_1}{}\quad 1$$

$$\langle p(t),(u^1_{xx})\quad p(t)\rangle$$

$$+\sum_{i=2}^{I}\left[\frac{\partial F_i}{\partial W}\langle p(t),D\varpi_i\rangle\right]$$

The term in the brackets () defines an invertible linear operator. The rest defines a finite rank linear operator. Therefore, from Theorem 3 we know that the index of F' is 0.

We know that $F^{-1}(\varpi_0)$ is a submersion at a regular value ϖ_0. Therefore, F' has empty cokernel at ϖ_0. Since the index of F' is 0, its kernel is also empty as well, which implies that F' is a linear isomorphism.

The rest of the proof is a simple application of inverse function theorem (Theorem 5) and Sard-Smale's theorem (Theorem 6). □

6. Further research

Our approach yields useful results. They are summarized as follows:

(i) We have assumed separability, which is widely used in the literature. With this assumption, most price vectors cannot support equilibria for the underlying economies. Therefore, it is safe to remove the set of prices that cannot support equilibria, without loss of any information about the set of equilibria. A key result in this paper is that, on this price space, the excess demand function is a Fredholm map with index zero. This makes the application of Sard-Smale's theorem possible.

(ii) The index of Fredholm maps can be explicitly calculated (see Proposition 4 above). This makes the set of equilibria more tractable mathematically (see Theorem 9 above).

It seems possible to extend the above results to the following cases:

1. A continuum of agents. The set of traders $\{1,\ldots,I\}$ is replaced by a unit interval. If agents in the economies are not `too diverse' in a proper sense, our results should be applicable.
2. More general utility functions. The assumption of separability is by no means necessary. Many utility functions, such as those which exhibit `habit formation' (see Heal and Ryder, 1973), or those which are called stochastic differential utility functions (see Duffie and Epstein, 1992), have similar regular properties.
3. Other topological spaces. For example, the main theorem also applies to Sobolev spaces.
4. The topological structure of the set of equilibria. This describes the set of solutions to simultaneous non-linear operators as the parameters of the economy vary. This set has been classified topologically in finite dimensional spaces (see Balasko, 1975, 1988 for complete markets, and Chichilnisky and

Heal, 1996, for incomplete markets) but not in infinite spaces. Using the techniques developed here it should be possible to obtain a complete characterization of the equilibrium manifold with infinite goods for both complete and incomplete markets.

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