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# The geometry of implementation: a necessary and sufficient condition for straightforward games*

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**Abstract.** We characterize games which induce truthful revelation of the players' preferences, either as dominant strategies (straightforward games) or in Nash equilibria. Strategies are statements of individual preferences on $R^n$. Outcomes are social preferences. Preferences over outcomes are defined by a distance from a bliss point. We prove that *g is straightforward if and only if g is locally constant or dictatorial* (*LCD*), i.e., coordinate-wise either a constant or a projection map *locally* for almost all strategy profiles. We also establish that: (i) If a game is straightforward and respects unanimity, then the map *g* must be continuous, (ii) Straightforwardness is a nowhere dense property, (iii) There exist differentiable straightforward games which are non-dictatorial. (iv) If a social choice rule is Nash implementable, then it is straightforward and locally constant or dictatorial.

## 1. A characterization of straightforward games

In classical forms of resource allocation for public goods,[1] efficiency requires accurate information about people's preferences. However, asking individuals to reveal their preferences can lead to a game in which the truth may or may not be the outcome. When is telling the truth the best strategy? Games in which players' best moves are to say the truth, are called *straightforward*. This

* The first versions of these results were completed in 1979, and they were then revised and extended in 1980 and 1981. Versions were circulated as Essex working papers under the titles "Incentives to Reveal Preferences", "Incentive Compatibility and Local Simplicity" and "A Necessary and Sufficient Condition for Straightforwardness". Research support from NSF Grants. SES 79-14050, 92-16028 and 91-10460 and the United Kingdom S.S.R.C. is gratefully acknowledged.

[1] Such as those proposed by Lindahl, Bowen and Samuelson.

paper gives necessary and sufficient conditions for a game to be straightforward.

In the search for straightforward games, certain points are obvious. If a player is a dictator, namely if the outcome is determined solely by her preferences, then she has no incentive to misrepresent those preferences. Likewise, if the outcome is constant, independent of the strategy chosen by the player, then there is no incentive to misrepresent either.

The insight offered in this paper is that these two simple and appealing cases serve as a basis for constructing *all* possible straightforward rules: within a certain family of single peaked preferences defined on the choice space $\Re^n$, *a rule is straightforward if, and only if, it is made up by "piecing together" constant rules and dictatorial rules*. Such rules are called *locally constant or dictatorial* (*LCD*), and they can be very different indeed from dictatorial or constant maps. However, *locally* they behave either like a constant function or like a dictatorial function (a projection) almost everywhere. LCD rules have a remarkably simple geometric structure.

The results presented here were developed between 1979 and 1981[2] and have been circulated widely since them. They are based on an intuitive geometric object: the preimage in strategy space of a given outcome. Our approach is unique in that all of our results are proven by reference to this geometric structure, and are valid for any Euclidean space. This geometric structure has proven to be fruitful elsewhere as well: it was adopted later by Saari [20] and by Rasmussen [19] in this volume, and it is also used in our results on "strategic dictators' in Chichilnisky and Heal [12] and in the results on strategic control in Chichilnisky [8]. We are able to do this because we show (in Theorem 1) that any straightforward game with a convex range (implied for example by respect of unanimity) must be continuous. We can therefore work with continuous maps between Euclidean spaces.

Though simple in concept, locally constant or dictatorial (LCD) maps can be quite complex: several examples are constructed here. LCD rules may satisfy desirable features: they can be continuous and anonymous[3] and also respect unanimity.[4] These are the three axioms proposed by Chichilnisky [6, 7] for characterizing desirable social choice rules.[5]

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[2] These results were widely circulated and presented at conferences and seminars from 1979 to 1982.

[3] A function $f(x_1, x_2, \ldots, x_n) = y$, $f: \Re^n \to \Re$, is anonymous if $f(x_1, x_2, \ldots, x_n) = f(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n})$ where $(\pi 1, \pi 2, \ldots, \pi n)$ is a permutation of the integers 1 to $n$. A social choice rule with this property does not discriminate between agents on the basis of their identity.

[4] A social choice rule $f(x_1, x_2, \ldots, x_n) = y$, $f: \Re^n \to \Re$, respects unanimity if $f(x, x, \ldots, x) = x \forall x$.

[5] Generally there exist no social choice rules satisfying Chichilnisky's three axioms, cf. Chichilnisky [6, 7]. In our case they exist because we restrict the domain of preferences, see also Chichilnisky and Heal [9]. These rules include various "generalized median" rules, such as those of Moulin [18], which are extensions of the median rule by the inclusion of non-existent voters, and those of Barberà, Gul and Stacchetti [2], elegantly defined by left- and right-coalition systems.

The attractive properties of these LCD rules are bought at a high price: there are very few such rules. Formally, LCD rules are nowhere dense in the space of continuous functions. Straightforward games are therefore not robust.[6]

In addition to simplicity, our characterization has clear advantages over alternative descriptions of straightforward rules in terms of medians and phantom voters[7]. LCD rules can be extended naturally to infinite populations, for which medians are not well-defined (see Heal [15]). Another advantage is that it provides a basis for analyzing the incentive-compatibility properties of Rawlsian rules. These have been widely studied and have the property that (locally) one individual[8] is dictatorial, the person who is in the worst position, and the rule is constant with respect to the preferences of all others. Therefore Rawlsian rules are straightforward.

Our results extend also to Nash equilibrium strategies. We show that being LCD is necessary and sufficient for truthful revelation to be a Nash equilibrium. So the apparently less demanding concept of Nash implementation in fact brings little in the way of greater generality.

We work with generalizations of single-peaked preferences,[9] in our case the indifference curves are families of ellipsoids. Choice spaces are linear subspaces of a Euclidean space.[10] The messages or strategies of the players are statements of their characteristics: these are either vectors in $R^{n+}$ (bliss points of the single peaked preferences), or alternatively, preferences over $R^n$. Outcomes, or payoffs, are vectors in $R^n$. Each player seeks through strategic behavior to attain an outcome as close as possible to his or her optimal outcome or bliss point, according to some distance on $R^n$.

The paper is organized as follows: the following section introduces the results and provides geometric examples. Section three proves rigorously the results on straightforwardness, and section four does likewise for Nash implementation with separable regular games.[11] The main part of the paper uses only geometric arguments; longer proofs are in the Appendix.

## 2. The geometry of implementation

This section gives an introduction to the subject by providing examples and simple geometric interpretations of the results.

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[6] A related fact was noted by Guesnerie and Laffont [13] in a different framework.

[7] Moulin [18] studies straightforwardness in terms of generalized median rules. His results apply only to one-dimensional choice spaces. Border and Jordan [4] work with so-called "phantom voters". They study voting rules where the population of voters is enlarged by imaginary or phantom voters.

[8] Not always the same individual, but always the individual occupying the position of being worst off.

[9] The same framework has been used by Moulin, Barberà Gul and Stacchetti, Barberà et al. [3], Border and Jordan [4] and van der Stel [23]. For an excellent recent review of this literature see van der Stel [23].

[10] Unlike Barberà et al., who work with discrete sets of choices.

[11] Separable regular games are defined fully below: separability means that $g:\Re^{mk} \to \Re^m$, $g = (g_1, g_2, \ldots, g_m)$. Regularity is a rank condition on the derivative of the game form.

We start with games where the players' characteristics are real numbers; later we consider more general cases. There are $k \geq 2$ players. Each player wishes to achieve an outcome in the real line which is as close as possible (in $\Re$) to this or her true "bliss point". Preferences are therefore represented by utility functions that are symmetric around a maximum value in $\Re$, the "bliss point". $S$ is the space of strategies and $A$ the space of outcomes. A game form $g: S^k \to A$ (also called a "rule") is a function which associates with each $k$-tuple of agents' strategies an outcome in $A$. A game $g$ respects unanimity if $g(p_1, \ldots, p_k) = \bar{y} \in A$ when for all $i = 1, \ldots, k$ the preferences $p_i$, have the same bliss point $\bar{y}$. The game $g$ just defined is called *straightforward* if the announcement of one's true characteristic is always a dominant strategy for each player.[12]

There is an equivalent expression for straightforward games, which we present here for clarity but which is unnecessary otherwise; one says that a game "implements" a social choice rule if the equilibria of the game are the outcomes of the social choice function applied to "true" individual preferences. Thus a straightforward game implements its game form $g$ as a social choice function. The notion of equilibrium can be based on dominant strategies, or be a Nash equilibrium: both are considered in this paper.

A game which is not straightforward is called manipulable: in such games players have incentives to lie.

### *2.1. Manipulable rules*

Standard games, such as average rules, are manipulable. It will help the intuition to see why. Consider the game as defined above, where

$$g: [0, 1]^2 \to [0, 1], \quad g(r_1, r_2) = \lambda r_1 + (1 - \lambda) r_2, \ \lambda \in [0, 1].$$

Figure 1 represents this game form: the slanted lines represent the hypersurfaces of the game form function $g$, $g^{-1}(r) = \{(r_1, r_2): g(r_1, r_2) = r\}$. The horizontal axis of the square are the strategies of player one; the vertical of player 2.

This game has an interesting characteristic: for any strategy $s_2$ of player two within the segment $S$, there exists a strategy for player one denoted $r(s_2)$, which can attain his/her preferred outcome or "bliss point" $r_1$, i.e., $g(r(s_2), s_2) = r_1$. It suffices to choose $r(s_2)$ so that $(r(s_2), s_2 \in g^{-1}(r)$. Furthermore, this optimal strategy for player one, $r(s_2)$, varies with $s_2$. Therefore, stating the true characteristic $r_1$ is generally *not* the best strategy for player one. In fact, it is easy to check that in general *this game has no dominant strategies*. This game is manipulable.

It is clear from the above discussion that, *to avoid manipulability, one must require that the optimal response* $r(s_2)$ *does not vary locally with* $s_2$. This implies, in the diagram of the game $g$, that the hypersurfaces $g^{-1}(r)$ are either (i) vertical, in which case $r(s)$ is always the same as $s$ varies within a neighborhood, or (ii) horizontal with $r(s) \equiv s$, so that $r$ cannot affect the outcome and $s$ has no incentive to lie, or else (iii) that the game $g$ has large indifference

[12] A strategy $r$ is *dominant* for player one if for all $s$ in $S$ $g(r, s) = \max_{t \in S}(g(t, s))$, according to player one's preference.

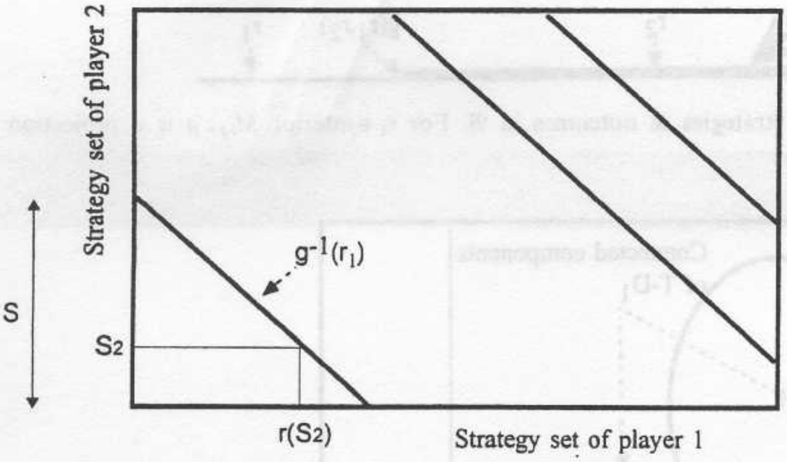


**Fig. 1.** The game $g(r_1, r_2) = \lambda r_1 + (1 - \lambda) r_2$, $\lambda \in [0, 1]$

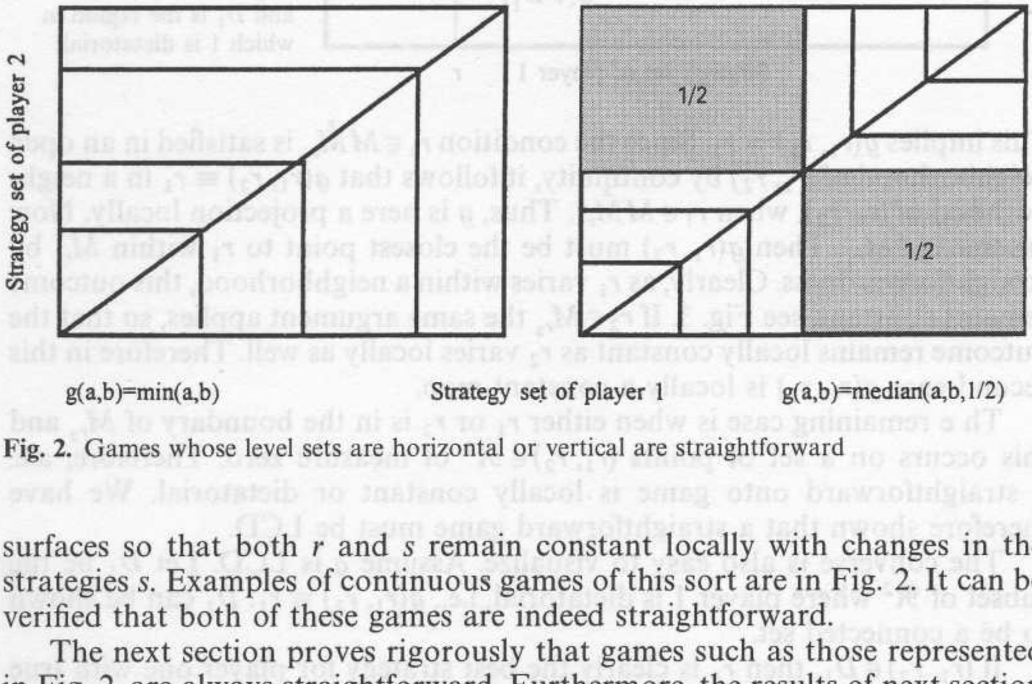


**Fig. 2.** Games whose level sets are horizontal or vertical are straightforward

surfaces so that both $r$ and $s$ remain constant locally with changes in the strategies $s$. Examples of continuous games of this sort are in Fig. 2. It can be verified that both of these games are indeed straightforward.

The next section proves rigorously that games such as those represented in Fig. 2, are always straightforward. Furthermore, the results of next section establish that all straightforward games are of this form. Why?

## 2.2. *Illustrating the results*

Why should straightforward rules be LCD? An intuitive argument is as follows.

Consider a game as above, $g : \Re^2 \to \Re$. Let $g$ be onto and straightforward. Somewhat surprisingly, in this case, $g$ must be continuous (see Theorem 1 below). Define now the manipulation set $M_{r_2}$: it is the set of outcomes which the first player can achieve when player two plays $r_2$. Then if the "true" bliss point of player 1, $r_1$, is in the interior of $M_{r_2}$, denoted $\mathring{M}_{r_2}$, $r_1$ is by definition achievable by player one with true characteristic $r_1$, by straightforwardness.

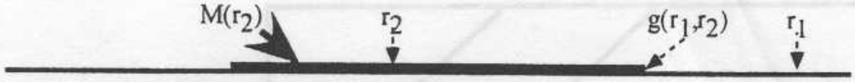


**Fig. 3.** 2 players with strategies & outcomes in $\Re$. For $r_1 \in$ interior $M_{r_2}$, $g$ is a projection: otherwise it is constant

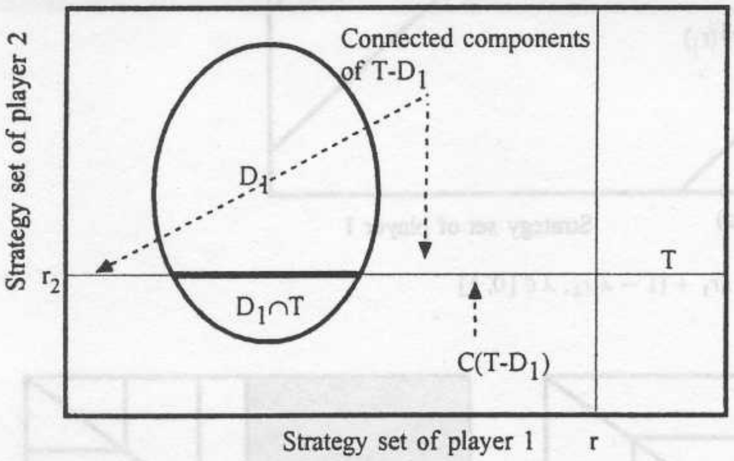


**Fig. 4.** The game is LCD, and $D_1$ is the region in which 1 is dictatorial

This implies $g(r_1, r_2) = r_1$. Since the condition $r_1 \in M\mathring{M}_{r_2}$ is satisfied in an open neighborhood of $(r_1, r_2)$ by continuity, it follows that $g(r_1, r_2) \equiv r_1$ in a neighborhood of $(r_1, r_2)$, when $r_1 \in M\mathring{M}_{r_2}$. Thus, $g$ is here a projection locally. Now assume $r_1 \notin M_{r_2}$. Then $g(r_1, r_2)$ must be the closest point to $r_1$ within $M_{r_2}$ by straightforwardness. Clearly, as $r_1$ varies within a neighborhood, this outcome remains constant, see Fig. 3. If $r_2 \in M_{r_2}$ the same argument applies, so that the outcome remains locally constant as $r_2$ varies locally as well. Therefore in this second case $g(r_1, r_2)$ is locally a constant map.

Th e remaining case is when either $r_1$ or $r_2$ is in the boundary of $M_{r_2}$ and this occurs on a set of points $(r_1, r_2) \in \Re^2$ of measure zero. Therefore, a.e. a straightforward onto game is locally constant or dictatorial. We have therefore shown that a straightforward game must be LCD.

The converse is also easy to visualize. Assume $g$ is LCD. Let $D_1$ be the subset of $\Re^2$ where player 1 is dictatorial, i.e., $g(r_1, r_2) \equiv r_1$. $D_1$ can be shown to be a connected set.

If $(r_1, r_2) \in D_1$, then $r_1$ is clearly the best strategy for player one with true characteristic $r_1$. Otherwise, if $r_1 \notin D_1$, let $T = \{(r, s) \in R^2 \colon s = r_2\}$ and $T - D_1$ be the part of $T$ not in $D_1$. By assumption, $g$ is locally constant on $T - D_1$ with respect to its first coordinate; since $g$ is continuous, $g$ must be constant on any connected component of $T - D_1$, $C(T - D_1)$. Assume that player one's true preference is $r_1$ and $(r_1, r_2) \in T - D_1$, see figure 4. Any point in this component of $T - D_1$ therefore gives the same outcome as $(r_1, r_2)$ so that there are no incentives to lie within this component of $T - D_1$. Furthermore, by continuity, $g(r_1, r_2) = r_1$ if[13] $(r_1, r_2) \in \partial\mathring{C}(T - D_1)$. In addition, the strategy $r' \neq r_1$ is also less preferable to $r_1$ if $r' \in \mathring{D}_1$ because $g(r, r_2) = r$ and is therefore further away from $r_1$ than is $g(r_1, r_2)$. Finally, if $(r', r_2)$ is in another connected component of $T - D_1$ where $g$ is locally constant, see figure 4,

[13] The boundary of a set $X$ is denoted $\partial X$.

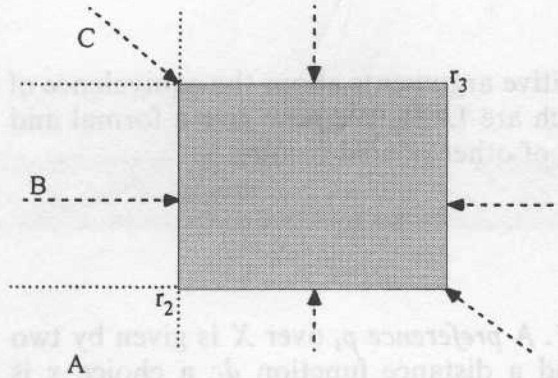


**Fig. 5.** Outcomes are 2-dimensional and there are 3 agents. Agents 2 and 3 announce $r_2$ and $r_3$: if agent 1 announces in the shaded area, 1 determines the median in each coordinate and is a dictator. For announcements by 1 outside the shaded area, the rule acts as a projection onto this area

$g(r', r_2)$ is still further away from $r_1$ than is $g(r_1, r_2)$ because it is at least as far as $g(r', r_2)$ where $r' \in \partial C(T - D_1)$. Therefore a rule which is LCD and onto is straightforward, as we wished to show.

Up to now the player's characteristics are real numbers. Now we consider two higher dimensional examples.

**Example 1.** Let $n = 2$ so that choices and bliss points are in $\Re^2$ and let the number of players $k = 2$. Define $g(r_1, r_2) = (r_{11}, r_{22})$, where $r_1 = (r_{11}, r_{12})$ and $r_2 = (r_{21}, r_{22})$.

Thus agent 1 is dictatorial in the first component, and agent 2 in the second. Clearly the rule $g$ is locally constant or dictatorial, and is straightforward. Agent 2's manipulation set is a vertical straight line through 1's announcement, and 1's is a horizontal straight line through 2's, and any announcement by 2 (or 1) leads to an outcome which is the horizontal (or vertical) projection of this into the vertical (or horizontal) line through 1's (or 2's) announcement.

**Example 2.** Now let $n = 2$ and $k = 3$ and $g(r_1, r_2, r_3) = (x_1, x_2, x_3)$ where $x_j = \text{median}\ (r_{1j}, r_{2j}, r_{3j})$.

This is a coordinate-wise median rule. Fig. 5 shows the manipulation set of agent 1, when 2 and 3 have announced $r_2$ and $r_3$ respectively. The manipulation set is shaded. If 1's announcement is contained in this, it is the median in both components and 1 is a dictator.

Consider regions A and B as indicated in Fig. 5. If $r_1$ is in region B, then $r_1$ has the median vertical component and $r_2$ the median horizontal component and the outcome is $(r_{21}, r_{12})$. Hence in region B, $g(., r_{-i})$ acts a horizontal projection onto the manipulation set, where $r_i$ is the vector $r$ with the $i$-th component deleted. In region A, $r_2$ has the median in both components and the outcome is $r_2$. Hence in A $g$ acts to project to the nearest point of manipulation set, $r_2$. In region C, $r_2$ has the horizontal and $r_3$ the vertical median, so the outcome is $(r_{21}, r_{32})$ and all points in C are mapped to the nearest corner of the shaded set. It is now routine to verify that $g(., r_{-i})$ acts elsewhere as shown in Fig. 5, which illustrates its action as a projection onto a convex set bounded by coordinate hyperplanes.

## 3. Main results

In the previous section we gave intuitive arguments about the equivalence of straightforward rules and rules which are LCD. We now give a formal and general statement of that result and of other related results.

### *3.1. Notation and definitions*

Let $X$ be the choice space, $X = \Re^{n+}$. A *preference* $p_i$ over $X$ is given by two objects: a "bliss" point $y^i$ in $X$, and a distance function $d_i$: a choice $x$ is preferred to another $z$ if $x$ is closer than $z$ to the bliss point $y^i$, i.e., $d_i(x, y^i) < d_i(z, y^i)$. The distance $d_i(x, y)$ is given by $\sum_{j=1}^{n} m_j(x_j - y_j)$, where $(m_j)$ is a strictly positive vector in $\Re^n$ (i.e., $d$ is not degenerate). The indifference surfaces of $p_i$ are then convex ellipsoids with center at the bliss point $y^i$ and axes parallel to the coordinate axes.

The *space of strategies or messages* $S$ is either (i) $S = \Re^{n+}$, in which case each message in $\Re^{n+}$ is interpreted as a statement of an agent's preferred outcome, or (ii) $S = P$, where $P$ is the space of all preferences (distances and bliss points) defined above. Thus either $(\Re^n)^k$ or $P^k$ is the space $S^k$ of strategy profiles for $k$ players. Since a preference in $P$ is uniquely identified by its bliss point and its metric[14], $P \approx \Re^{2n+}$. The space of outcomes $A$ is $\Re^{n+}$ in either case.

A *game form* is now a map $g: S^k \to A$. Continuity of $g$ is defined with respect to the usual topology of Euclidean spaces. When the game form $g$ can in principle take any value in $\Re^n$, $g$ is called onto.

A *game* is given by a game form as above, and a family $\{p_i\}$ of preferences over outcomes, designated by matrices $M_i \in P$, $i = 1, \ldots, k$.

The symbol $(m_i, m_{-i})$ denotes a message or strategy profile in $S^k$, with its i-th. component equal to $m_i$ in $S$ and where $m_{-i}$ is a $k-1$ vector of strategies for players other than $k$.

A strategy profile $(m_1^*, \ldots, m_k^*)$ is a *dominant strategy equilibrium* if for all $i = 1, \ldots, k$ and $m_{-i}$, the outcome $g(m_i^*, m_{-i})$ is preferred to the outcome $g(m_i, m_{-i})$, for all $m_i \in P$, according to player $i$'s preference $p_i$.

[14] The inner product $\langle , \rangle_i$ defining the metric $d_i$ can be identified by a matrix $M$, such that $\langle x, y \rangle_i = \langle Mx, y \rangle$, where $\langle , \rangle$ denotes the standard inner product in $\Re^n$. The matrix $M$ is symmetric and also positive definite (in order to be non-degenerate). We shall assume that preferences are separable, in the sense that if any two choices $x$ and $z$ differ only on their $k$-th coordinates, $x$ is preferred to $z$ when the coordinate $x_k$ is preferred to the coordinate $z_k$ (i.e., $x_k$ is closer to the $k$-th coordinate of the bliss point $y$ than $z_k$). In this case the matrix $M$ is diagonal, and we can write

$$d_i(x, y) = \|x - y\|_i = \langle m, (x - y) \rangle = \sum_{j=1}^{n} m_j(x_j - y_j),$$

where $m$ is a positive vector in $\Re^n$. We assume $m$ is strictly positive, since otherwise the metric would be degenerate. A preference is thus completely identified by a bliss point $y_i$ in $\Re^{n+}$, and by a strictly positive vector $m \in \Re^n$.

A game $g$ is *straightforward* if $(r_1, \ldots, r_k)$ is a dominant strategy equilibrium for players with characteristics $(r_1, \ldots, r_k)$ in $S^k$, i.e., truthful messages about characterstics are dominant strategies for each player.

The *pre-manipulation set* of $m_{-i}$ is the set

$$N_{m_{-i}} = \{(m_i, m_{-i}) \in S^k \colon m_i \in S\}.$$

A function $f: \Re^s \to \Re$ is *locally constant or dictatorial* (*LCD*) if it is continuous, and for almost all[15] $x$ in $\Re^s$, there exists a neighborhood $N_x \subset \Re^s$ with

$$f/N_x = \text{constant}$$

or

$$f/N_x(y) \equiv y_d, \quad \text{for some } d \in \{1, \ldots, k\}, \text{ for all } y \text{ in } N_x.$$

For higher dimensional domains and ranges, a function $f: (\Re^m)^k \to \Re^m$ is called *separable* if the $j$-th. coordinate of the image depends only on the $j$-th. coordinates of the arguments, i.e.,

$$f(x_1^1, \ldots, x_m^1, \ldots, x_1^k, \ldots, x_m^k) = f_1(x_1^1, \ldots, x_1^k), \ldots, f_m(x_m^1, \ldots, x_m^k).$$

A function $f: (\Re^m)^k \to \Re^m$ is called *LCD coordinate-wise or LCD*, if it is separable, $f = (f_1, \ldots, f_m)$, and if each $f_i$ is LCD for $i = 1, \ldots, m$.

We consider here game forms which are not necessarily onto: their images are *linear subsets* of $A = \Re^n$, i.e., $g(S^k) = \Re^s \subset \Re^n$ with $s \leq n$, or $g(S^k) = \{(x_1, \ldots, x_s) \in \Re^s \colon a_i \leq x_i \leq b_i, \text{ each } i\}$. Such games do not necessarily respect unanimity even if they are straightforward. The next result will show that any such straightforward game $g: S^k \to A$ is LCD, where the strategy space $S$ is either $\Re^n$ or the space of preferences $P$ defined above. Furthermore, being LCD is also sufficient for straightforwardness.

Let $X$ be a subset of a topological space $Y$. Then $X$ is *residual* if it is a countable intersection of open and dense subsets of $Y$. A residual set in a complete normed space is always dense.

### *3.2. Lemmas*

The following are simple but useful properties of straightforward games.

**Lemma 1.** *A straightforward game $g: (\Re^n)^k \to \Re^n$ respects unanimity if and only if it is onto.*

*Proof.* In the Appendix. ■

**Lemma 2.** *If a game $g: \Re^2 \to \Re$ is straightforward and onto, then the outcome $g(r_1, r_2)$ is contained in the segment $[r_1, r_2]$.*

*Proof.* In the Appendix. ■

**Lemma 3.** *If $k = 2$, $g$ is straightforward and its image is a segment $[a, b] \subset \Re$, then either the outcome $g(r_1, r_2)$ is in the segment $[r_1, r_2]$, or else $g(r_1, r_2)$ is in the*

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[15] "*Almost all*" indicates for all points in $\Re^s$, except possibly on a subset of Lebesgue measure zero in $\Re^s$.

*boundary of the segment $[a, b]$ denoted $\partial[a, b]$. Furthermore, if the strategy $r_1$ is not in $[a, b]$, then the outcome $g(r_1, r_2)$ is the same as the outcome $g(x, r_2)$, where $x$ is the closest point to $r_1$ in $[a, b]$.*

*Proof.* In the Appendix. ■

Now define the *manipulation set* $M_{r-i}$, i.e., the set of outcomes that can be obtained by player $i$ when all other players have announced a vector of messages $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k) = r_{-i}$ in $\Re^{k-1}$. $M_{r-i} = \{y\colon y = \phi(r, r_{-i}), r \in \Re\}$

**Lemma 4.** *If $g\colon \Re^k \to \Re$ is straightforward and the strategy $r_i$ is in $M_{r_{-i}}$ then the outcome $g(r_1, \ldots, r_k) = r_i$. If $r_i \notin M_{r_{-i}}$, then the outcome $g(r_1, r_k)$ is the closest point to $r_i$ in the boundary of $M_{r_{-i}}$, denoted $\partial M_{r_{-i}}$.* In particular, $g(\Re^n)^k$ is closed.

*Proof.* In the Appendix. ■

*3.3. Straightforward games with a convex image are continuous*

**Theorem 1.** *If the choice space is one-dimensional, $g\colon \Re^k \to \Re$ is straightforward and its image $g(\Re^k)$ is convex, then $g$ is continuous. In particular, if $g$ is straightforward and respects unanimity, then is it continuous.*

*Proof.* In the Appendix. ■

**Remark 1.** Not all straightforward games are continuous. Figure 6 gives an example of a discontinuous straightforward game with a non-convex range.

It is clear that the game in Fig. 6a is straightforward for any pair $(r_1, r_2)$ in the interior of one of the shaded areas. When the characteristics of the players $(r_1, r_2)$ are either $r_1 = \frac{1}{2}$ or $r_2 = \frac{1}{2}$, it is easy to see that $g$ is straightforward also, since the outcomes $\frac{1}{4}$ and $\frac{3}{4}$ are equi-distant from $\frac{1}{2}$. Saying the truth is thus a dominant strategy for both players. The example in Fig. 6a can be generalized to produce straightforward games with a large, even countably infinite, number of discontinuities. Figure 6b shows a rule which is LCD, discontinuous and *not* straightforward. So without continuity the equivalence of straightforwardness and being LCD does not hold, although straightforwardness and a convex range together imply continuity (Theorem 1).

*3.4. Straightforward rules and LCD rules*

**Lemma 5.** *Let $\phi\colon P^k \to A$ be a locally constant or dictatorial (LCD) rule. Then $\phi$ is straightforward.*

*Proof.* The proof is in the Appendix. ■

We can now state formally the main result of this paper:

**Theorem 2.** *Let $g\colon S^k \to A$ be a game form. Then $g$ is straightforward if and only if $g$ is locally constant or dictatorial.*

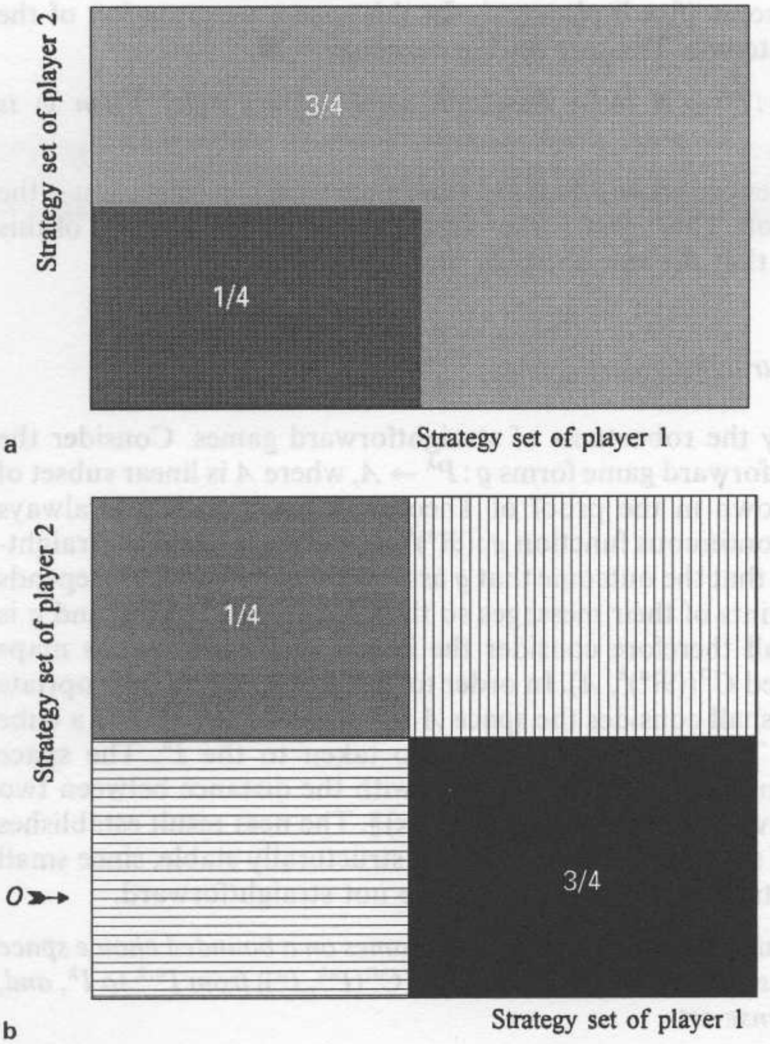


**Fig. 6.** **a** The game $g:[0,1]^2 \to [0,1]$ is defined by $g(r_1, r_2) = 1/4$ if $r_1 \leq 1/2$ and $r_2 \leq 1/2$; $g(r_1, r_2) = \frac{3}{4}$ otherwise. **b** In this case, $g(x_1, x_2) = 0$ for $0 \leq x_1, x_2 \leq \frac{1}{2}$, $=1$ for $\frac{1}{2} \leq x_1, x_2 \leq 1$, $=\frac{1}{4}$ for $0 < x_1 \leq \frac{1}{2}$ and $\frac{1}{2} \leq x_2 \leq 1$, and $=\frac{3}{4}$ for $\frac{1}{2} < x_1 \leq 1$ and $0 \leq x_2 < \frac{1}{2}$

*Proof.* The proof is in the Appendix. Sufficiency is clearly established by Lemma 5 above. The formal proof builds on the intuitive arguments of the previous section. ■

An immediate implication of our main result is that smooth straightforward rules onto $\Re^n$ are coordinate-wise dictatorial:

**Corollary 1.** *If $g:(\Re^n)^k \to \Re^n$ is straightforward and differentiable, and its image is $\Re^n$, then $g$ is coordinate-wise dictatorial. In particular, $g$ is not anonymous.*

*Proof.* Since by Theorem 1, $g$ is LCD, by differentiability each coordinate must be either constant everywhere, or a projection. If the map $g$ was constant in one coordinate, $g$ could not be onto $\Re^n$. Therefore $g$ must be dictatorial coordinate-wise, implying that each coordinate $j$ is identically equal to the

$j$-th. coordinate of some (fixed) player, $d_j$. In this case a permutation of the agents alters the outcome. This proves the corollary. ■

**Corollary 2.** *Let $\phi: P^k \to A$ be a Rawlsian social choice rule. Then $\phi$ is straightforward.*

*Proof.* Rawlsian rules are those which maximize the utility of one agent – the agent who is worst off. This agent is therefore dictatorial. The identity of this agent may vary, so that the rule s locally dictatorial. ■

### *3.5. Robustness of straightforward games*

We shall now study the robustness of straightforward games. Consider the family of all straightforward game forms $g: P^k \to A$, where $A$ is linear subset of $\mathfrak{R}^n$. As we have shown in the proof of Theorem 2, each such $g$ is always representable as a continuous function $g: (\mathfrak{R}^n)^k \to A$. This is because straightforwardness implies that the outcome that $g$ assigns to messages in $P^k$ depends only on the bliss points of their messages so that Theorem 1 applies and $g$ is continuous. We shall therefore consider the family of all continuous maps $f: (\mathfrak{R}^n)^k \to A$, denoted $C^0((\mathfrak{R}^n)^k, A)$. In order to give this space an appropriate norm topology, we shall consider the space $A$ to be bounded, i.e., $A$ is a cube in $\mathfrak{R}^n$, denoted $I^n$. The message space is also taken to the $I^n$. The space $C^0((\mathfrak{R}^n)^k, A)$ is given the sup norm topology, with the distance between two functions $f$ and $g$ given by $\sup_{x \in \mathfrak{R}^{nk}} \|f(x) - g(x)\|$. The next result establishes that the property of straightforwardness is not structurally stable, since small deformations of a straightforward function are not straightforward.

**Theorem 3.** *The family of non-straightforward games on a bounded choice space is a residual set of the space of continuous maps $(C^0(I^{nk}, I^n))$ from $I^{n,k}$ to $I^k$, and, in particular, is a dense set.*

*Proof.* The proof is in the Appendix.

## 4. Nash implementation

The following result analyzes the problem of Nash implementation in cases where preferences $P$, messages $M$ and outcomes $A$ are one dimensional. Consider a game form $g: M^k \to A$, where both the message space $M$ and the outcome space $A$ are one dimensional, $M = A = I$, $I$ the unit interval in $\mathfrak{R}$. We shall require $g$ to be a $C^{k+1}$ map,[16] and to satisfy a *regularity condition* (1) defined as follows.

**Regularity condition:** Let $\eta = (\eta_1, \ldots, \eta_{k-j})$ denote a non-empty subset of $k - j$ integers in $\{1, \ldots, k\}$. Define the map $g_\eta: M^k \times I \to \mathfrak{R}^{k-j+1}$ by

$$g_\eta(m, b) = (Dg_{\eta 1}(m) + g(m), \ldots, Dg_{\eta k-j}(m) + g(m), g(m))$$

[16] For related results on implementation of smooth maps, see Laffont and Maskin [17].

for all $(m, b) \in M^k \times I$ where $Dg_{\eta 1}$ is the matrix of first partial derivatives of $g$ with respect to $\eta_1$. Let $\Delta_\eta$ be the "diagonal" subset

$$\Delta_\eta = \{(x_1, \ldots, x_{k-j+1}) \in \Re^{k-j+1}: x_i = x_j \forall i = 1, \ldots, k-j+1\}.$$

Then $g$ must be transversal to $\Delta\eta$, i.e.

$$g_\eta \pitchfork \Delta_\eta, \ \forall \eta \subset \{1, \ldots, k\}. \tag{1}$$

Condition (1) applies to any $(m, b) \in M^k \times I$ such that all $k-j+1$ coordinates of its image in $R^{k-j+1}$ under $g_\eta(m, b)$ are equal. This is equivalent to $g(m) = b$ and $Dg_{\eta_1}(m) + b = b, \ldots, Dg_{\eta_{k-j}}(m) + b = b$, or equivalently $g(m) = b$ and $Dg_{\eta_1}(m) = \ldots = Dg_{\eta_{k-j}}(m) = 0$. In this case condition (1) implies that the gradient $Dg_\eta(m, b)$ has rank $k-j$ at such points, i.e., at points $(m, b) \in M^k \times I$ mapping into the diagonal of $R^{k-j+1}$.

A game form $g$ satisfying (1) is called *regular*. Note that condition (1) does not imply the gradient $Dg(x) \neq 0$ for all $x \in I^k$. The following lemma establishes that (1) is satisfied for a generic set of $C^{k+1}$ games:

**Lemma 6.** *Consider the family G of all $C^{k+1}$ maps $\{g: I^k \to I\}$, endowed with the $C^{k+1}$ sup topology. Condition (1) is satisfied on a residual (and in particular dense) set of G.*

*Proof.* The proof is omitted due to space constraints, but can be obtained from the authors. It is in a Columbia Business School Working Paper with the same name as this article.

Note that the genericity of the condition (1) on $C^{k+1}$ games, does not imply that the set of rules to be Nash implemented is a generic set. The following theorem proves that only very special rules will be Nash-implementable. The reason for this is that a large class of regular games will implement the same rule.

**Theorem 4.** *Let $\phi$ be a continuous social choice rule, $\phi: P^k \to A$ where $P = A = I$, the unit interval in $\Re$, and $k \geq 2$*[17]*. Let $M = I$ be the message space consisting of statements on bliss points of individual preferences. If the rule $\phi$ is Nash implementable by a regular game $g: M^k \to A$, then $\phi$ is locally constant or dictatorial (LCD).*

*Proof.* This is proved in the Appendix. ■

From Theorem 4 we obtain the following result, which is valid for any euclidean space of outcome or message spaces:

**Theorem 5.** *(1) Let $\phi: P^k \to A$ be a rule which is Nash implementable by a separable regular game*[18] *g. Then $\phi$ is straightforward and LCD. (2) If*

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[17] The space of preferences $P$ is here identified with the space of outcomes $A = I$ by assigning to each preference $p \in P$ its (unique) bliss point $b$ in $I$. Therefore the space $P$ is given the standard topology of $I$ in which two preferences $p_1$, $p_2$ close it their bliss points $b_1$, $b_2$ are close in $I$. Continuity of $\phi$ refers here to this topology on $P$.

[18] A separable game $g: I^{km} \to I^m$, $g = (g_1, \ldots, g_m)$ is said to be regular if each component function $g_i$, $i = 1, \ldots, m$, is regular.

*$\phi : P^k \to A$ is straightforward and regular, then $\phi$ is LCD and (of course) Nash implementable by a separable regular game.*

*Proof.* First note that if a rule $\phi$ is Nash implementable by a separable game $g$, then $\phi$ is separable. Therefore by Theorem 4 if $\phi$ is Nash implementable by a separable regular game $g = (g_1, \dots, g_m)$, then $\phi$ is separable, i.e., $\phi = (\phi_1, \dots, \phi_m)$, and each $\phi_i$ is implementable by a regular game. Therefore, each $\phi_i$ is LCD, which implies, by definition, that $\phi$ is LCD.

The converse is immediate: any straightforward rule $\phi$ is obviously Nash implementable by the game form it defines. Therefore we can apply Theorem 2 and this proves that $\phi$ must be locally constant or dictatorial. This completes the proof. ■

## 5. Conclusions

Being locally constant or dictatorial has been shown to be a necessary and sufficient condition for straightforwardness. This is an intuitively appealing result: it is clear that constant rules or dictatorial rules are straightforward. Our result says that the only straightforward rules are those obtained by "patching together" in a continuous fashion rules where are locally constant or locally dictatorial in each coordinate. However, such rules may be quite complex: many LCD games are simultaneously continuous and anonymous, and also respect unanimity. Hence they satisfy the axioms introduced by Chichilnisky [6], and subsequently used by others, for characterizing ethically acceptable social choice rules. Such axioms are not satisfied in general without restrictions on preferences, see, e.g., Chichilnisky [6] and Chichilnisky and Heal [9].

Moreover, as the "patching together" of constant and dictatorial rules can be done continuously but not smoothly, it immediately follows that if one requires smoothness, the rules must be either constant or dictatorial on each component. If onto, the rule must be dictatorial coordinate-wise, and thus cannot be anonymous.

We therefore have a rather simple and intuitive characterization of the possibilities for straightforward implementation, which shows that only a rather special type of rule is straightforward, and that the class of such rules is not robust. We have also shown that being locally constant or dictatorial a necessary and sufficient condition for implementability via the Nash equilibria of a separable regular game. So with separable rules, even though Nash implementable rules is much less demanding, the relaxation of the implementation concept (from straightforwardness to Nash) does not change the results.

## A. Appendix

### *A.1. Proofs of results on straightforwardness*

#### *A.1.1. Proofs of lemmas*

**Lemma 1.** *A straightforward game $g : (\mathfrak{R}^n)^k \to \mathfrak{R}^n$ respects unanimity if and only if it is onto.*

*Proof.* If $g$ respects unanimity then the image of $g$ covers $\Re^n$. The converse is also immediate. Assume $g$ is straightforward and onto; for any $r \in \Re^n$ let $r = g(r_1, \ldots, r_k)$, i.e., $r$ is attainable by player 1 (by announcing $r_1$) when the other players announce a $k-1$ vector $(r_2, \ldots, r_k)$. Since $r$ is the best outcome for player 1 with true characteristic $r$, it follows by straightforwardness that by stating the truth, player 1 must be able to attain $r$, i.e., $g(r_1, \ldots, r_k) = r \Rightarrow g(r, r_2, \ldots, r_k) = r$.

Iterating this procedure, one obtains $g(r, \ldots, r) = r$, i.e., $g$ respects unanimity. ■

**Lemma 2.** *If a game $g: \Re^2 \to \Re$ is straightforward and onto, then the outcome $g(r_1, r_2)$ is contained in the segment $[r_1, r_2]$.*

*Proof.* Being onto and straightforward, $g$ respects unanimity by lemma 1, so that $g(r_2, r_2) = r_2$. It follows that for any $r_1$ $g(r_1, r_2)$ must be at least as preferable to player one with true characteristic $r_1$ as is $r_2$, so $g(r_1, r_2)$ is closer to $r_1$ than is $r_2$. Since this is also true for player 2, then $g(r_1, r_2) \in [r_1, r_2]$. ■

**Lemma 3.** *If $k = 2$, and the game form $g$ is defined on $\Re$, is straightforward and its image is a segment $[a, b] \subset \Re$, then either the outcome $g(r_1, r_2)$ is in the segment $[r_1, r_2]$, or else $g(r_1, r_2)$ is in the boundary of the segment $[a, b]$ denoted $\partial[a, b]$. Furthermore, if the strategy $r_1$ is not in $[a, b]$, then the outcome $g(r_1, r_2)$ is the same as the outcome $g(x, r_2)$, where $x$ is the closest point to $r_1$ in $[a, b]$.*

*Proof.* Consider first the case when $[r_1, r_2] \cap [a, b] = \phi$. In this case by straightforwardness the outcome of $(r_1, r_2)$ must be the closest point to $r_1$ and to $r_2$ in $[a, b]$, i.e., a point in $\partial[a, b]$. Secondly, consider the case of $[a, b] \subset [r_1, r_2]$. Then the lemma is obviously true, by the assumptions on $g$. Thirdly, note that if $r_1 \in [a, b]$ and $r_2 \in [a, b]$, then obviously $g(r_1, r_2) \in [r_1, r_2]$. Finally, suppose that both $r_1$ and $r_2$ are in $[a, b]$. Then assume $g(r_1, r_2) = r$. By straightforwardness, $g(r, r_2) = r$ and similarly $g(r, r) = r$. Hence the restriction of $g$ on $[a, b]$, $g/[a, b]$, is onto and lemma 2 applies. ■

Now recall the definition of the *manipulation set* $M_{r-i}$, i.e., the set of outcomes that can be obtained by player $i$ when all other players have announced a vector of messages $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k) = r_{-i}$ in $\Re^{k-1}$, $M_{r-i} = \{y: y = \phi(r, r_{-i}), r \in \Re\}$.

**Lemma 4.** *If the game form $g: (\Re^n)^k \to \Re^n$ is straightforward and the strategy $r_i$ is in $M_{r-i}$ then the outcome $g(r_1, \ldots, r_k) = r_i$. If $r_i \in M_{r_{-i}}$, then the outcome $g(r_1, \ldots, r_k)$ is the closest point to $r_i$ in the boundary of $M_{r_{-i}}$, denoted $\partial M_{r_{-i}}$.* In particular, $g(\Re^n)^k$ is closed.

*Proof.* This is immediate. If $r_i \in M_{r-i}$, then $r_i$ is attainable by the i-th. player, given the other messages $r_{-i}$. Then by straightforwardness $g(r_1, \ldots, r_i, \ldots, r_k) = r_i$. If $r_i \notin M_{r_{-i}}$, then by straightforwardness $g(r_1, \ldots, r_i, \ldots, r_k)$ must be the best attainable by player $i$ with characteristics $r_i$. Thus $g(r_1, \ldots, r_i, \ldots, r_k)$ must be the closest point to $r_i$ in $M_{r_{-i}}$, and thus in $\partial M_{r_{-i}}$. ■

**Theorem 1.** *If the choice space is one-dimensional, $g: \Re^k \to \Re$ is straightforward and its image $g(\Re^k)$ convex, then $g$ is continuous. In particular, if $g$ is straightforward and respects unanimity, then it is continuous.*

*Proof.* The strategy of the proof is as follows. We consider first the case of 2 players, and then extend the result to any number by induction. In the case of two players, we deal first with the case in which neither $r_1$ nor $r_2$ are in the image of $\Re^2$ under $g$, $g(\Re^2)$. In the remaining case we show that the graph of $g$ is closed, and hence that it is continuous.

Consider first the case of two players, $k = 2$. Assume first that neither $r_1$ nor $r_2$ are in $g(\Re^2)$ and let the sequence $(r_1^h, r_2^h) \to (r_1, r_2)$. If $[r_1, r_2]$ does not intersect the image of $g$, then there exists $N$ such that $[r_1^h, r_2^h]$ will not intersect $g(\Re^2)$ either $\forall h > N$. Thus, for $h > N$, $g(r_1^h, r_2^h) \equiv x$, where $x$ is a point in $\partial g(\Re^2)$, by Lemma 3. Since the outcome $g(r_1, r_2)$ is the same point $x$ in this case, this proves continuity.

If $r_1$ is not in the image $g(\Re^2)$ but $r_2$ is, then $g(r_1, r_2) = g(x, r_2)$ by lemma 3, for some $x$ in $g(\Re^2)$, so we consider next the case where *both* $r_1$ *and* $r_2$ *are in the image* of $g$. We show that in this case the graph of $g$ is closed.

Let $r_1$ and $r_2$ be in $g(\Re^2)$ and $r_1 \neq r_2$. By Lemma 4 if $r_1 \in M_{r_{-1}}$, the manipulation set of agent 1, then $g(r_1, r_2) = r_1$ and $g$ is continuous. Now assume that $r_1 \notin M_{r_{-1}}$. Since $\lim_h r_i^h = r_i$, $i = 1, 2$, $\exists N$ such that for $h > N$, $r_1^h \neq r_2^h$, and

$$g(r_1^h, r_2^h) = a \in \partial M_{r_{-1}}.$$

Since $r_2^h \to r_2$, by lemma 4, $g(r_1^h, r_2) = a$ for $h > N$, as $a$ will also be the nearest point in $M_{r_{-1}}$ to $r_1^h$. This implies that

$$\lim_h g(r_1^h, r_2) = a.$$

We now claim that both points $a$ and $g(r_1, r_2)$ must be at the same distance from $r_1$. To see this note that if $a$ is nearer then player one could obtain a better outcome by stating $r_1^h$, whereas if $g(r_1, r_2)$ is nearer then player 1 with true preference $r_1^h$ has an incentive to misrepresent and state $r_1$ as her true preference. So either $g(r_1, r_2) = a$ or the two points $g(r_1, r_2)$ and $a$ are equidistant from $r_1$ on opposite sides. However, $g(r_1, r_2) \in [r_1, r_2]$ and $g(r_1^h, r_2) \in [r_1^h, r_2]$ by Lemma 3, implying that $g(r_1, r_2)$ and $a$ cannot be on opposite sides of $r_1$ so that

$$g(r_1, r_2) = \lim_h g(r_1^h, r_2) = a.$$

Since $\lim_h (r_1^h, r_2) = \lim_h g(r_1^h, r_2^h) = g(r_1, r_2)$, we have shown that the limits of points in the graph of $g$, i.e., points of the form $\lim_h (r_1^h, r_2^h, g(r_1^h, r_2^h))$, are always in the graph of $g$, since they are equal to $(r_1, r_2, g(r_1, r_2))$. Thus, the map $g$ has a *closed graph*, which is equivalent to being continuous. This completes the case in which both $r_1$ and $r_2$ are in the image of $g$ and $r_1 \neq r_2$.

Now we consider the case where $r_1 = r_2$. First assume that $r_1 = r_2$, and they are both in the *interior* of the image $g(\Re^2)$. Then since $g(r_1^h, r_2^h) \in [r_1^h, r_2^h]$ for $h > N$, by lemma 3, we can pick sequences $r_1^h$ and $r_2^h$ such that without loss of generality $r_1^h < r_1 = r_2 < r_2^h$ so that $\lim_h g(r_1^h, r_2^h) = r_1 = g(r_1, r_2)$ and continuity is again ensured.

If $r_1 = r_2$ and they are both in the *boundary* of $g(\Re^2)$ then for $h > N$ we can assume without loss of generality one of three cases:

either $r_1^h \in \partial g(\Re^2)$ and $r_2^h \in g(\Re^2)$ in which case $g(r_1^h, r_2^h) = r_1^h$ by lemma 3; or both $r_1^h$ and $r_2^h$ are not in $g(\Re^2)$, in which case by lemma 3

$$g(r_1^h, r_2^h) = r_1$$

or, finally, both $r_1^h$ and $r_2^h$ are in $g(\mathfrak{R}^2)$, in which case

$$g(r_1^h, r_2^h) \in [r_1^h, r_2^h].$$

In any of these three cases, $\lim_h g(r_1^h, r_2^h)$ is $g(r_1, r_2)$ so that as before, the graph of $g$ is closed, and thus $g$ is continuous. This completes the proof of continuity for two players.

The argument for two players is clearly valid when there are $k > 2$ players, provided one is restricted to sequences in $\mathfrak{R}^k$ in which only two (the $i$-th. and $j$-th.) players vary their messages, i.e., sequences of the form

$$\lim(r_1, \ldots, r_j^h, r_{j+1}, \ldots, r_i^h, r_{i+1}, \ldots, r_k) = (r_1, \ldots, r_k) \in \mathfrak{R}^k.$$

We shall now prove continuity of $g$ by induction, assuming continuity when up to $k-1$ players are allowed to vary their messages.

**Inductive hypothesis:** $g$ is continuous in any $k-1$ of its arguments.

Let $(r_1^h, \ldots, r_k^h) \to (r_1, \ldots, r_k)$, and denote by $r^h$ the value $g(r_1^h, \ldots, r_k^h) = r^h$. If for all $i$, $r_i^h$ is in player i's manipulation set, i.e., $r_i^h \in M_{r_{-i}}^h$, then all $r_i^h$'s must be identical, since in this case

$$g(r_1^h, \ldots, r_k^h) = r_i^h \quad \text{for all } i,$$

by Lemma 4. Continuity is assured in this case, since $r_i^h \to r_i$ for all $i$, and $g(r_1, \ldots, r_k) = r_i \; \forall i$. Otherwise, if some message $r_i^h$ is *not* in $M_{r_{-i}}^h$, then by Lemma 4, for $h > N$

$$r^h = g(r_1^h, \ldots, r_{i-1}^h, r_i, r_{i+1}^h, \ldots, r_k^h) = g(r_1^h, \ldots, r_k^h),$$

because by choosing $N$ sufficiently large we can ensure that $r_i$ and $r_i^h$ will be as close as desired. The problem is therefore reduced to one in which only $k-1$ messages are allowed to vary, and by the induction hypothesis this completes the proof. ■

**Lemma 5.** *Let $\phi: P^k \to A$ be a locally constant or dictatorial rule. Then $\phi$ is straightforward.*

*Proof.* Consider first the case where $P = A = \mathfrak{R}$.

The strategy of the proof is as follows: We consider three exhaustive and exclusive cases. The first case is when agent $j$'s true preference $\bar{p}_j$ is such that individual $j$ is a dictator when telling the truth, i.e. $g(\bar{p}_j, p_{-j}) = \bar{b}_j$, the bliss point of $\bar{p}_j$. The second case is when there is no preference that $j$ can announce such that he or she is a dictator, i.e. $\forall p_i, g(p_i, p_{-j}) \neq \bar{b}_j$. Finally, the third case is when $\bar{p}_j$ is such that $j$ is not a dictator when telling the truth, but can become a dictator by misrepresentation, i.e. there exists a $p'_j \neq \bar{p}_j$ such that $g(p'_j, p_{-j}) = b'_j$ where $b'_j$ is the bliss point of $p'_j$. In each of these three cases, we show that truthful revelation is a dominant strategy.

Assume that individuals' true preferences are given by the profile $(\bar{p}_1, \ldots, \bar{p}_k)$. We wish to prove that $\bar{p}_j$ is a dominant strategy for the $j$-th. individual.

Define $D_j \subset P^k$ to be the region where $j$ is dictatorial. For any $k-1$ tuple of strategies of agents other than $j$, denoted $p_{-j}$, there are three mutually exclusive and exhaustive cases:

(a) $(\bar{p}_j, p_{-j}) \in D_j$, i.e., $\phi(\bar{p}_j, p_{-j}) = \bar{b}_j$, where $\bar{b}_j$ is the bliss point of $\bar{p}_j$. Agent $j$ is a dictator when telling the truth.

(b) $g(p_j, p_{-j}) \neq b_j$ for any $p_j \in P$, i.e. $(p_j, p_{-j}) \notin D_j$ for any $p_j \in P$. Agent $j$ is never dictatorial.

(c) $g(\bar{p}_j, p_{-j}) \neq \bar{b}_j$, but there exists some $p'_j \in P$ such that $g(p'_j, p_j) \neq b'_j$ (i.e. $(\bar{p}_j, p_{-j}) \notin D_j$ but $\exists p'_j \in P$ s.t. $(p'_j, p_{-j}) \in D_j$). Agent $j$ is not dictatorial when telling the truth, but can become "dictatorial by misrepresentation."

In case (a) it is obvious that $\bar{p}_j$ (i.e. the truth) is a dominant strategy for $j$. These cases are illustrated in Fig. 7. In case (b) let

$$\Psi(p_{-j}) = \{p_j \in P: (p_j, p_{-j}) \in P^k\}$$

In the set $\Psi(p_{-j})$ only $p_j$ varies: by assumption, $g$ is not dictatorial with dictator $j$ in this set. Hence $g$ fails to be constant with respect to $p_j$ only on a set of measure zero in $\Psi(p_{-j})$. By continuity and because we are in case (b) the set $\Psi(p_{-j})$ has only one connected component. This implies $g(., p_{-j})$ must be a constant on all of $\Psi(p_{-j})$, which implies that the true message $\bar{p}_j$ is as good a strategy as any in $P$ for the $j$-th. individual.

In case (c), consider the set $\tilde{D}(p_{-j})$ of strategies in $P$ for the $j$-th. individual

$$\tilde{D}(p_{-j}) = \{q_j \in P: (q_j, p_{-j}) \in D_j\}.$$

$\tilde{D}(p_{-j})$ is thus the set of strategies that make $j$ dictatorial within $\Psi(p_{-j})$. Consider now the strategy $\tilde{p}_j$ in $\tilde{D}(p_{-j})$ which is nearest in terms of the distance $d(\ldots)$ in $\Re$ to the true preference $\bar{p}_j$ (see Fig. 7), and let

$$d_0 = d(\tilde{p}_j, \bar{p}_j) = \min_{q_j \in \tilde{D}(p_{-j})} (d(q_j, \bar{p}_j)), \tag{2}$$

Note that $d_0 \neq 0$ by the construction of case (c). Outside of $\tilde{D}(p_{-j})$, $g(., p_{-j})$ is constant on any connected subset of $\Psi(p_{-j})$, by continuity. Hence it is

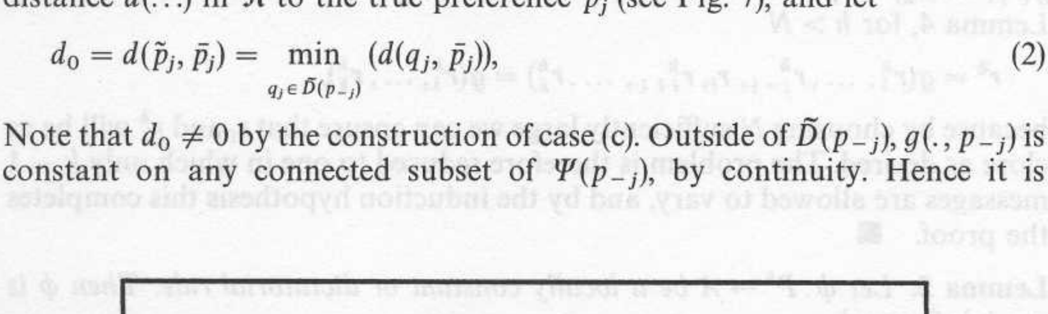


Fig. 7. An illustration of the proof that LCD rules are straightforward

constant on the following connected set $S$, a set of outcomes not attainable by player $j$ given the strategies $p_{-j}$:

$$S = \{q \in P: d(q, \bar{p}_j) \leq d_0\} \subset P,$$

so that by (2) for all $q$ in $S$, $g(q, p_{-j}) = g(\tilde{p}_j, p_{-j})$ by continuity. In particular, $g(\bar{p}_j, p_{-j}) = g(\tilde{p}_j, p_{-j})$. Now, if $q_j \in \tilde{D}(p_{-j})$ is such that $d(\bar{p}_j, q_j) > d_0$, obviously on the set $\Psi(p_{-j})$, $q_j$ is a strategy with a less desirable outcome for individual $j$ than $\tilde{p}_j$, and therefore, also a less desirable outcome than the truth $\bar{p}_j$.

Assume without loss of generality that $\tilde{p}_j < \bar{p}_j$ and consider the set

$$F_1 = \{q \in P: q \leq \min\{p: p \in \tilde{D}(p_{-j})\}\}$$

Then $g(., p_{-j})$ is a constant map for all strategies in $F_1$. Since $d(F_1, \bar{p}_j) \geq d(\tilde{D}(p_{-j}), \bar{p}_j) \geq d_0$, and for $q_j \in F_1$, $g(q_j, p_j)$ is equal by continuity to $g(\tilde{q}_j, p_{-j})$ for some $\tilde{q}_j$ in $\tilde{D}(p_{-j})$, it follows that strategies in $F_1$ have a less desirable outcome than the truth $\bar{p}_j$. Now consider $F_2 = \{q \in P: q \geq \max\{q \in S\}\}$. This is a connected set on which $g$ is constant. By continuity the outcome is equal to $g(\tilde{p}_j, p_{-j}) = g(\bar{p}_j, p_{-j})$. This completes the proof of straightforwardness when $P = C = I$.

Consider now $P = A = \Re^n$. Then, by definition since $g$ is locally constant or dictatorial it is separable, i.e.

$$g(p_1, \dots, p_k) = g_1(b_1^1, \dots, b_k^1), \dots, g_n(b_1^n, \dots, b_k^n),$$

where $b_i^j$ denotes the $j$-th. component of individual $i$'s bliss point. Since the arguments given above apply to each $g_i: (\Re^k) \to \Re$, $i = 1, \dots, n$, it follows that each $g_i$ is straightforward, so that $g$ is straightforward. This completes the proof of the proposition. ■

A.1.2. Proof of theorem 2

We now prove the main result of the paper, the equivalence of straightforwardness to being locally constant or dictatorial. The sufficiency of being LCD was of course established in Lemma 5, so that what remains is the necessity of being LCD.

**Theorem 2.** *A map $g$ is straightforward if and only if it is locally constant or dictatorial* (*LCD*).

The strategy of the proof is as follows.

1. First we prove that being LCD is necessary for straightforwardness when the choice space is one dimensional, so that a game is a map from $\Re^k$ to $\Re$. In this case all metrics on the choice space agree and so preferences are characterized fully by their bliss points.
2. We then extend the result to higher dimensional cases. First we do this just for the case in which agent's strategies consist solely of announcing bliss points, and show that in this case any straightforward rule must be separable in the sense that the $i$-th. coordinate of the outcome depends only on the $i$-th. coordinates of the agents' strategies. In this case each coordinate function is a map from $\Re^k$ to $\Re$ and the results of the first case can be applied.
3. Next we analyze the case in which agents' strategies involve announcing the metrics of preferences as well as their bliss points. We show that in

this case the outcome of any straightforward rule must be unaffected by the metric announced, and so this case reduces to the previous one.

*Step 1.* Case $n = 1$, strategies are bliss points only.

Note that in the one dimensional case $P = \Re$, because preferences are statements of bliss points only, since all (non trivial) distances in $\Re$ are equivalent to the Euclidean distance. Assume that $g: S^k \to A$ is straightforward. By Theorem 1, $g$ is continuous.

Let $(m_i, m_{-i})$ be a profile in $P^k$. Consider first the case where $m_i$ is in the interior of the manipulation set $\mathring{M}_{m_{-i}}$. Then it follows by straightforwardness that

$$g(m_i, m_{-i}) = m_i. \tag{3}$$

Since $g$ is continuous, if $m^1_{-i}$ is a small variation of $m_{-i}$, $m_i$ is also in $\mathring{M}_{m^1_{-i}}$, so that

$$g(m_i, m^1_{-i}) = m_i \tag{4}$$

for all $m^1_{-i}$ in a neighborhood $U_{m_{-i}}$ of $m_{-i}$ in $S^{k-1}$.

Similarly, continuity of $g$ implies that if $m^1_i$ is a small variation of $m_i$, $m^1_i$ is in $\mathring{M}_{m^1_{-i}}$, so that (3) and (4) are also satisfied in a neighborhood of $m_i$.

We have therefore proven that for any profile $(m_i, m_{-i}) \in S^k$, if $m_i \in \mathring{M}_{m_{-i}}$, then $g$ is dictatorial with dictator $i$ in a neighborhood $W(m_i, m_{-i})$ of $(m_i, m_{-i})$ in $S^k$.

Consider now the case of a profile $(m_i, m_{-i}) \in S^k$ where $m_i \notin M_{m_{-i}}$ for all $i = 1, \ldots, k$.

In that case for any $i$

$$g(m_i, m_{-i}) = m \neq m_i. \tag{5}$$

Furthermore, by straightforwardness $m$ is the best that the $i$-th. player can obtain, so that $m \in M_{m_{-i}}$ minimizes the distance between $m_i$ and $M_{m_{-i}}$ in $A = \Re$. It follows that for $m^1_i$ a small variation of $m_i$,

$$g(m^1_i, m_{-i}) = m, \tag{6}$$

since $m$ will also minimize the distance in A between $m^1_i$ and $M_{m_{-1}}$, see Fig. 8.

Therefore, $g$ is a constant on a neighborhood $V$ of $m_i$, within the pre-manipulation set $N_{m_{-i}}$, see Fig. 9.

Since the same argument is valid for all $i = 1, \ldots, k$, it follows that $g$ is locally constant in a whole neighborhood $V(m_i, m_{-i}) \subset S^k$.

We have therefore proven that for any profile $(m_i, m_{-i}) \in S^k$, the map $g$ is LCD whenever either $m_i \in \mathring{M}_{m_{-i}}$ for some $i$, or else $m_i \notin \mathring{M}_{m_{-i}}$, for all $i$. Note that since $g$ is dictatorial with dictator $i$ when $m_i \in \mathring{M}_{m_{-i}}$, then for any profile $(m_j, m_{-j}) \in P^k$, at most *one* message say $m_j$ can be in the interior of the manipulation set determined by the others, $\mathring{M}_{m_{-j}}$. For any given profile $(m_i, m_{-i}) \in S^k$, there are therefore three exclusive and exhaustive cases:

(a) $m_j \in \mathring{M}_{m-j}$ for some $m_j$, a component of $(m_i, m_{-i})$ or

(b) $m_j \notin M_{m_{-j}}$ for all components $m_j$ of $(m_i, m_{-i})$ or

(c) $m_j \in \partial M_{m_{-j}}$ for some component $m_j$ of $(m_i, m_{-i})$.

As seen above, in case (a) the $j$-th. player is a dictator in a neighborhood of $(m_i, m_{-i})$, because the property $m_j \in \mathring{M}_{m_{-j}}$ is open in $S^k$. Therefore $g$ is LCD at

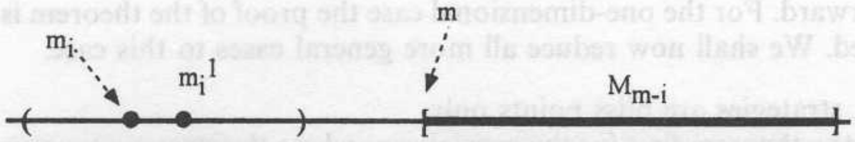


**Fig. 8.** Proof that $g$ is straightforward $\Leftrightarrow$ it is LCD, when $S$ is one dimensional. By straightforwardness the outcome $m$ is locally independent of $m_i$

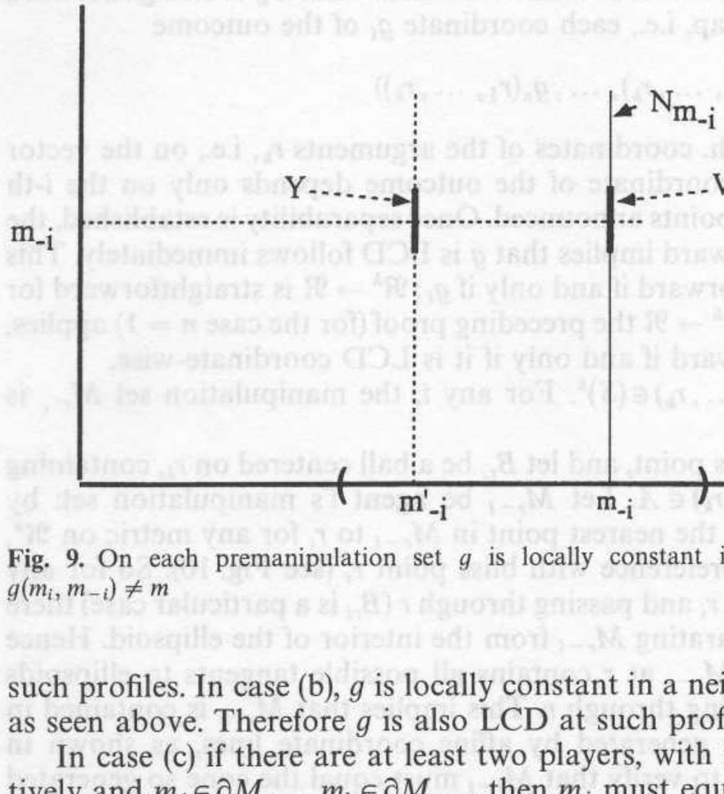


**Fig. 9.** On each premanipulation set $g$ is locally constant in a neighborhood of $m_i$ if $g(m_i, m_{-i}) \neq m$

such profiles. In case (b), $g$ is locally constant in a neighborhood of $(m_i, m_{-i})$, as seen above. Therefore $g$ is also LCD at such profiles.

In case (c) if there are at least two players, with messages $m_j$, $m_k$ respectively and $m_j \in \partial M_{m_{-j}}$, $m_k \in \partial M_{m_{-k}}$, then $m_j$ must equal $m_k$, since

$$m_j = g(m_j, m_{-j}) = g(m_k, m_{-k}) = m_k$$

by straightforwardness. Now, the space of profiles in $S^k$ having at least two coordinates equal is a set of measure zero in $S^k$. Therefore case (c) is contained in a set of measure zero in $S^k$ if $m_j \in \partial M_{m_{-j}}$ and $m_k \in \partial M_{m_{-k}}$ and $j \neq k$.

Now consider the last case in (c), where $m_j \in \partial M_{m_{-j}}$ for only one $m_j$ in $(m_i, m_{-i})$. The manipulation sets $M_{m_{-j}}$ are closed intervals.[19] Therefore, the map assigning to each $k-1$ profile $m_{-j}$ in $S^{k-1}$ the strategy $m_j$ in the boundary $\partial M_{m_{-j}}$ (a set consisting of two points in $\Re^1$) is the union of two continuous real valued maps on $S^{k-1}$. Since the graph of each of these maps is a set of measure zero in $S^k$, it follows that the set of profiles $(m_k, m_{-i})$ such that $m_j \in \partial M_{m_{-j}}$ for one $j \in \{1, \ldots, k\}$ has measure zero in $S^k$. Since this is true for each $j$, it follows that the set of profiles in case (c) have measure zero. This completes the proof that straightforward rules are LCD in the one-dimensional case. The converse has already been proven in Lemma 5: all LCD rules

[19] This is established in Lemma 4.

are straightforward. For the one-dimensional case the proof of the theorem is thus completed. We shall now reduce all more general cases to this case.

*Step 2.* $n > 1$, strategies are bliss points only.

We prove the theorem first for the special case *where the strategy space is just* $\mathfrak{R}^n$, *i.e.*, $S = \mathfrak{R}^n$ and the game form is $g:(\mathfrak{R}^n)^k \to A$, where $A$ is a linear subset of $\mathfrak{R}^n$. This is the case in which agents' strategies are just the bliss points of preferences, and do not involve the statement of metrics.

The strategy of this proof is as follows: we show that if $g$ is straightforward then $g$ is a separable map, i.e., each coordinate $g_i$ of the outcome

$$g(r_1, \dots, r_k) = (g_1(r_1, \dots, r_k), \dots, g_n(r_1, \dots, r_k))$$

depends only on the $i$-th. coordinates of the arguments $r_k$, i.e., on the vector $r_1^i, \dots, r_k^i$. So the $i$-th coordinate of the outcome depends only on the $i$-th coordinates of the bliss points announced. Once separability is established, the result that $g$ straightforward implies that $g$ is LCD follows immediately. This is because $g$ is straightforward if and only if $g_i : \mathfrak{R}^k \to \mathfrak{R}$ is straightforward for all $i$, and for each $g_i : (\mathfrak{R})^k \to \mathfrak{R}$ the preceding proof (for the case $n = 1$) applies, so that $g$ is straightforward if and only if it is LCD coordinate-wise.

Consider now $(r_1, \dots, r_k) \in (S)^k$. For any $i$, the manipulation set $M_{r_{-i}}$ is a convex set in A.[20]

Let $r_i \in \mathfrak{R}^n$ be $i$'s bliss point, and let $B_{r_i}$ be a ball centered on $r_i$, containing the point $r = g(r_1, \dots, r_k) \in A$. Let $M_{r-i}$ be agent $i$'s manipulation set: by straightforwardness $r$ is the nearest point in $M_{r-i}$ to $r_i$ for any metric on $\mathfrak{R}^n$, corresponding to any preference with bliss point $r_i$ (see Fig. 10). So for any ellipsoid $E_{ri}$ centered on $r_i$ and passing through $r$ ($B_{ri}$ is a particular case) there exists a hyperplane separating $M_{r-i}$ from the interior of the ellipsoid. Hence the set of supports to $M_{r-i}$ at $r$ contains all possible tangents to ellipsoids centered on $r_i$ and passing through $r$. This implies that $M_{r-i}$ is contained in a cone based at $r$ and generated by affine coordinate lines, as shown in Fig. 10a. Now it is easy to verify that $M_{r-i}$ must equal the cone so generated rather than being strictly contained in it. This follows from straightforwardness and the fact that the outcome is by assumption independent of the metrics of agents' preferences. This is illustrated in figure 10b, where the nearest point in the manipulation set $M_{r-i}$ to the bliss point $r_i$ depends on the metric around $r_i$. This would not be true if $M_{r-i}$ were generated by affine coordinate lines, as in the first panel.

It is clear that in this case, as shown in Fig. 10a, the $k$-th. coordinate of the outcome $r = g(r_1, \dots, r_k)$ depends *only* on the $k$-th. coordinates of the vectors $r_i$. Since this is true for all $k$, the separability of $g$ is established. This completes the proof for the case $g:(\mathfrak{R}^n)^k \to A$, because each $g_i$ must be LCD. In

[20] This can be easily seen from the arguments in Laffond [16], derived from those of Valentine [22], because $M_{r_{-i}}$ is, in this case, the manipulation set that would obtain if the $i$-th player's preference was required to have a Euclidean distance function (and any bliss point), which is the case studied by Laffond, see, e.g., his Theorem 1 and Lemmas 5 and 8. Note that the condition of anonymity of $g$ is not required in these proofs, and that the proof applies for $r_{-i}$ in $(R^n)^{k-1}$ as in our case.

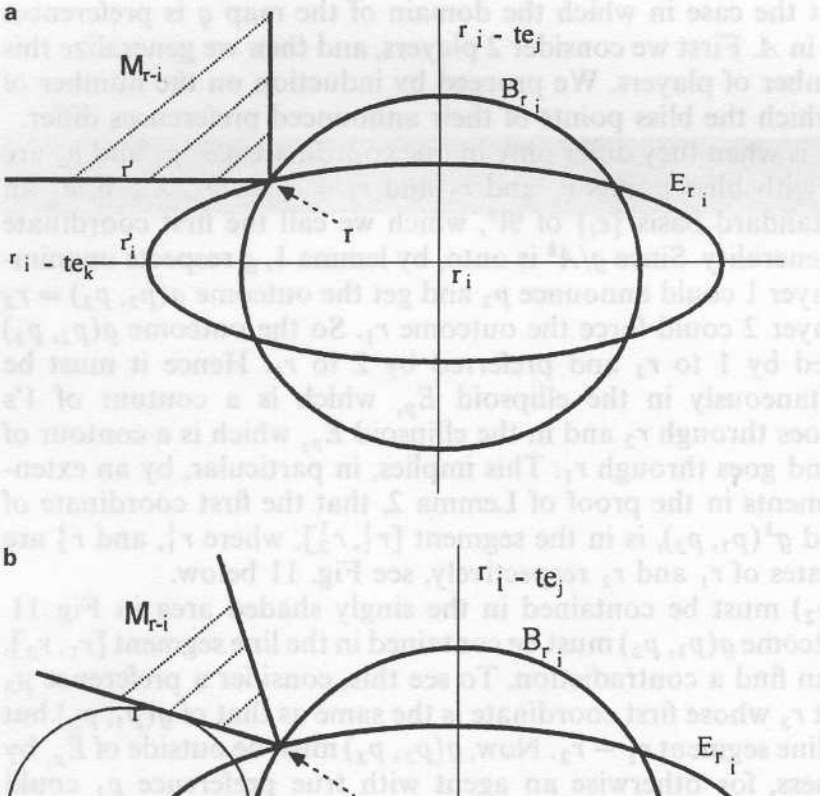


**Fig. 10a, b.** Proof that $g$ is straightforward $\Leftrightarrow$ it is LCD. Case when $n > 1$. The manipulation set is the cone generated by the coordinate lines through $r$

particular, it shows that if $g: (\Re^n)^k \to A$ is straightforward, then $g$ is also continuous, since each component $g_i$ is.

*Step 3.* Strategies include metrics.

Finally, we consider the case where choices are in $\Re^n$, but strategies are preferences in $P$, so that $g$ is a map defined on bliss points and metrics, $g: (P^n)^k \to A$, $A$ a linear subset of $\Re^n$.

We break this step into two sub steps. In the first of these, we show that when restricted to preferences with bliss points in $A$, $g$ only depends on these bliss points and *not* on the metrics announced by the players. This implies that $g/A^k$ is actually a map from $A^k \subset (\Re^n)^k$ into $\Re^n$; therefore, we can apply the results of the previous case to show that $g$ is separable, and thus the proof is completed for strategy profiles in $A^k$.

In the second sub step, we then show the proof is also valid for profiles in $(\Re^n)^k$ outside $A^k$, and the proof is completed.

Consider first the case in which the domain of the map $g$ is preferences with bliss points in $A$. First we consider 2 players, and then we generalize this to any finite number of players. We proceed by induction on the number of coordinates in which the bliss points of their announced preferences differ.

The first case is when they differ only in one coordinate, i.e., $p_1$ and $p_2$ are two preferences with bliss points $r_1$ and $r_2$ and $r_1 = r_2 + \lambda e_1$, $\lambda > 0$, $e_1$ an element of the standard basis $\{e_i\}$ of $\Re^n$, which we call the first coordinate without loss of generality. Since $g/A^k$ is onto, by lemma 1, $g$ respects unanimity. Therefore player 1 could announce $p_2$ and get the outcome $g(p_2, p_2) = r_2$ and similarly player 2 could force the outcome $r_1$. So the outcome $g(p_2, p_2)$ must be preferred by 1 to $r_2$ and preferred by 2 to $r_1$. Hence it must be contained simultaneously in the ellipsoid $E_{p_1}$ which is a contour of 1's preference and goes through $r_2$ and in the ellipsoid $E_{p_2}$ which is a contour of 2's preferences and goes through $r_1$. This implies, in particular, by an extension of the arguments in the proof of Lemma 2, that the first coordinate of $g(p_1, p_2)$, denoted $g^1(p_1, p_2)$, is in the segment $[r_1^1, r_2^1]$, where $r_1^1$, and $r_2^1$ are the first coordinates of $r_1$ and $r_2$ respectively, see Fig. 11 below.

Thus, $g(p_1, p_2)$ must be contained in the singly shaded area in Fig. 11. Note that the outcome $g(p_1, p_2)$ must be contained in the line segment $[r_1, r_2]$, otherwise one can find a contradiction. To see this, consider a preference $p_3$ with a bliss point $r_3$ whose first coordinate is the same as that of $g(p_1, p_2)$ but which lies in the line segment $r_1 - r_2$. Now, $g(p_3, p_2)$ must be outside of $E_{p_1}$ by straightforwardness, for otherwise an agent with true preference $p_1$ could announce $p_3$ and obtain an outcome preferable to $g(p_1, p_2)$. By the arguments of the previous paragraph, we must also have $g(p_3, p_2) \in E_{p_3}$ which is the ellipsoid centered on $r_3$ and through $r_2$, for otherwise a first agent with true preference $p_3$ could do better by announcing a preference $p_2$ and obtaining $p_2$ by respect of unanimity. This is true for any metrics and so for any ellipsoids $E_{p_1}$ and $E_{p_3}$ respectively centered on $r_1$ and $r_3$ and through $r_2$. This immediately establishes a contradiction, as is clear from Fig. 11. This contradiction cannot be established if $g(p_1, p_2) \in [r_1, r_2]$.

Now, if $g(p_1, p_2)$ were to vary within $[r_1, r_2]$ as the two players vary their metrics (keeping their bliss points $r_1$ and $r_2$ fixed) then obviously, the outcome could be manipulated by an appropriate choice of metric. Thus, when the bliss points of $p_1$ and $p_2$ differ in one coordinate only, $g$ must be independent of the metric announced.

Furthermore, note that when $r_1$ and $r_2$ differ in one coordinate only, if $g(p_1, p_2) \in [r_1, r_2]$, the interior of the segment $[r_1, r_2]$, then $g(p_1, p_2)$ is a constant map for all $p_1$ and $p_2$ with bliss points $r'_1$ and $r'_2$ in the segment $[r_1, r_2]$: this follows from the characterization of straightforward games as LCD maps for the case $n = 1$, and the fact that as the outcome is neither $r_1$ nor $r_2$ so that neither player is dictatorial.

We now make the following *inductive hypothesis*:

**Inductive assumption.** (1) If $p_1$ and $p_2$ are two preferences whose bliss points $r_1$ and $r_2$ differ at most in m-1 coordinates, then the outcome $g(p_1, p_2)$ depends only on the bliss points $r_1$ and $r_2$ and, furthermore, (2) if for some coordinate $j$, $g^j(p_1, p_2)$ is in the interior of $[r_1^j, r_2^j]$, then $g$ is constant for all $(p_1, p_2)$ whose bliss points are in the box $B[r_1, \ldots, r_2]$ determined by $r_1$ and $r_2$, i.e.,

$$\{b \in \Re^n \colon \forall i, r_a^i \leq b^i \leq r_b^i, i = 1, \ldots, n \text{ where } \{a, b\} = \{1, 2\} \text{ or } \{2, 1\}\}.$$

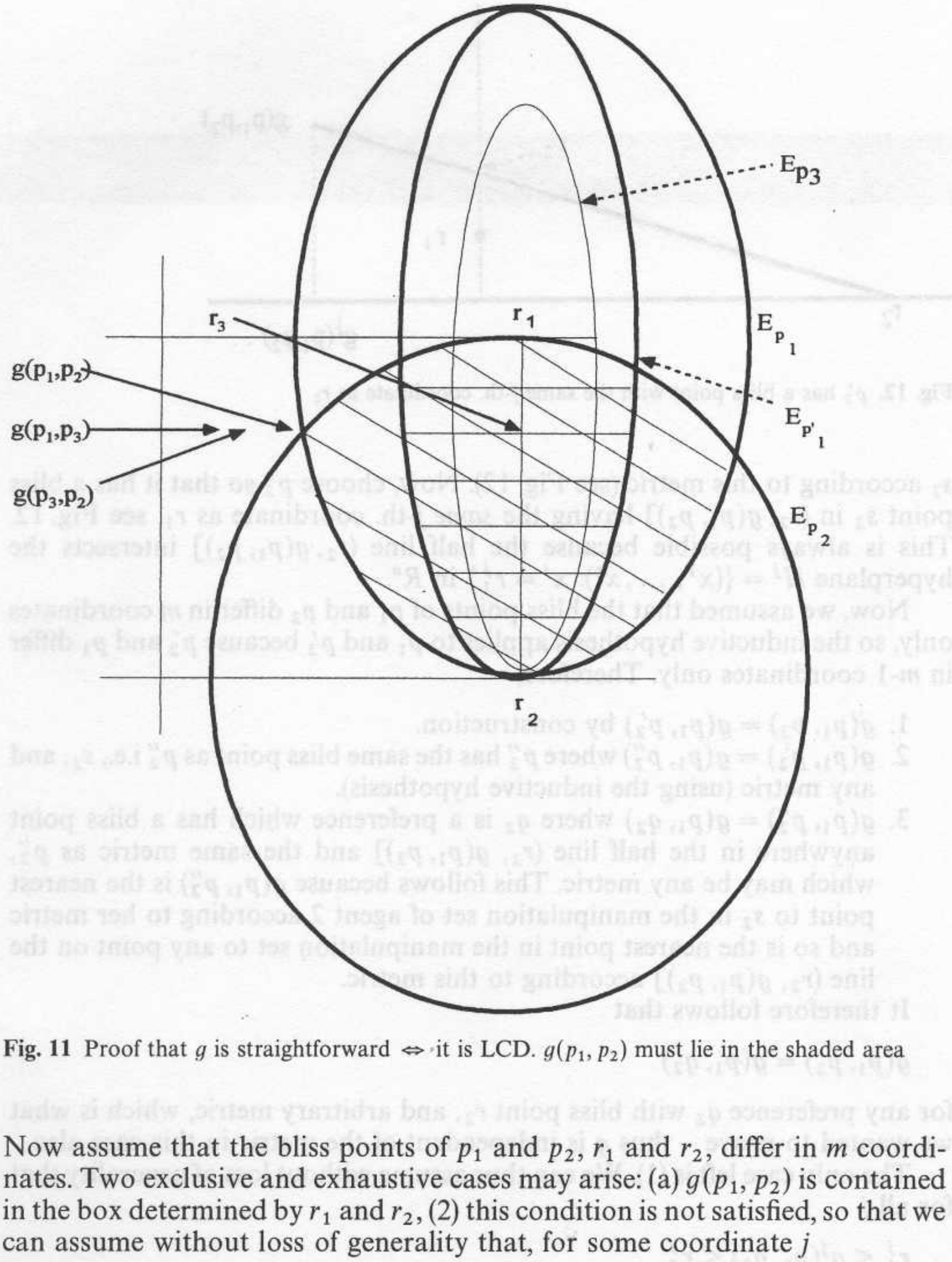


**Fig. 11** Proof that $g$ is straightforward $\Leftrightarrow$ it is LCD. $g(p_1, p_2)$ must lie in the shaded area

Now assume that the bliss points of $p_1$ and $p_2$, $r_1$ and $r_2$, differ in $m$ coordinates. Two exclusive and exhaustive cases may arise: (a) $g(p_1, p_2)$ is contained in the box determined by $r_1$ and $r_2$, (2) this condition is not satisfied, so that we can assume without loss of generality that, for some coordinate $j$

$$g^j(p_1, p_2) > r_1^j > r_2^j,$$

see Fig. 12 below

In case (2), note that for all preferences $p_2'$ with bliss points $s_2$ in the half line $(r_2, g(p_1, p_2)]$ and same metric $m_2$ as $p_2$, we have

$$g(p_1, p_2) = g(p_1, p_2');$$

this follows from the fact that as $g(p_1, p_2)$ is the nearest point to $r_2$ in the manipulation set of agent 2 according her metric, it is also the nearest point to

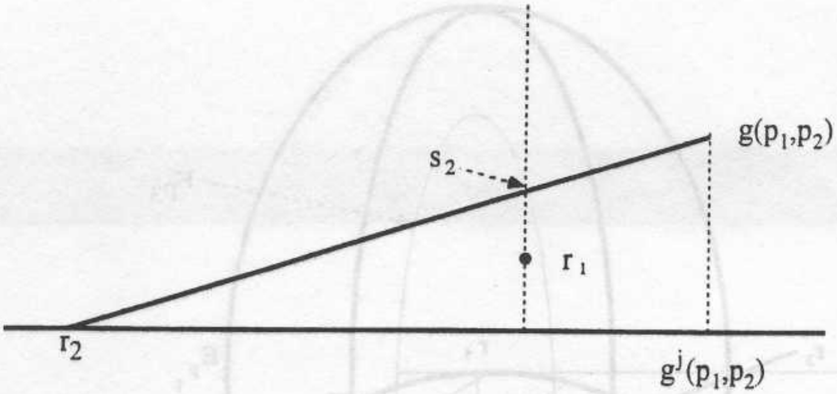


**Fig. 12.** $p_2'$ has a bliss point with the same $j$-th. coordinate as $r_1$

$s_2$ according to this metric (see Fig. 12). Now, choose $p_2'$ so that it has a bliss point $s_2$ in $(r_2, g(p_1, p_2)]$ having the *same* $j$-th. coordinate as $r_1$, see Fig. 12. This is always possible because the half line $(r_2, g(p_1, p_2)]$ intersects the hyperplane $H^j = \{(x^1, \ldots, x^n): x^j = r_1^j\}$ in $R^n$.

Now, we assumed that the bliss points of $p_1$ and $p_2$ differ in $m$ coordinates only, so the inductive hypothesis applies to $p_1$ and $p_2'$ because $p_2'$ and $p_1$ differ in $m$-1 coordinates only. Therefore,

1. $g(p_1, p_2) = g(p_1, p_2')$ by construction.
2. $g(p_1, p_2') = g(p_1, p_2'')$ where $p_2''$ has the same bliss point as $p_2''$ i.e., $s_2$, and any metric (using the inductive hypothesis).
3. $g(p_1, p_2'') = g(p_1, q_2)$ where $q_2$ is a preference which has a bliss point anywhere in the half line $(r_2, g(p_1, p_2)]$ and the same metric as $p_2''$, which may be any metric. This follows because $g(p_1, p_2'')$ is the nearest point to $s_2$ in the manipulation set of agent 2 according to her metric and so is the nearest point in the manipulation set to any point on the line $(r_2, g(p_1, p_2)]$ according to this metric.

It therefore follows that

$$g(p_1, p_2) = g(p_1, q_2)$$

for any preference $q_2$ with bliss point $r_2$, and arbitrary metric, which is what we wanted to prove – thus $g$ is independent of the metric in this case also.

The only case left is (1). We can thus assume without loss of generality that for all $j$

$$r_1^j \leq g^j(p_1, p_2) \leq r_2^j,$$

Note that if for all $p_1, p_2$, one part of this inequality is an equality for all $j$, the result is automatically true because $g(p_1, p_2)$ is then in the boundary of the box determined by $r_1$ and $r_2$ so that its $h$-th coordinate depends only on the $h$-th coordinates of $r_1$ and $r_2$, i.e. it is separable. We can therefore assume a strict inequality for some $j$; without loss of generality, assume

$$r_1^j < g^j(p_1, p_2) < r_2^j.$$

The rest of the proof is simple: we show that we can alter $p_1$ and $p_2$ into $\bar{p}_1$ and $\bar{p}_2$ so that $\bar{p}_1, \bar{p}_2$ have bliss points which differ in $m-1$ coordinates only and

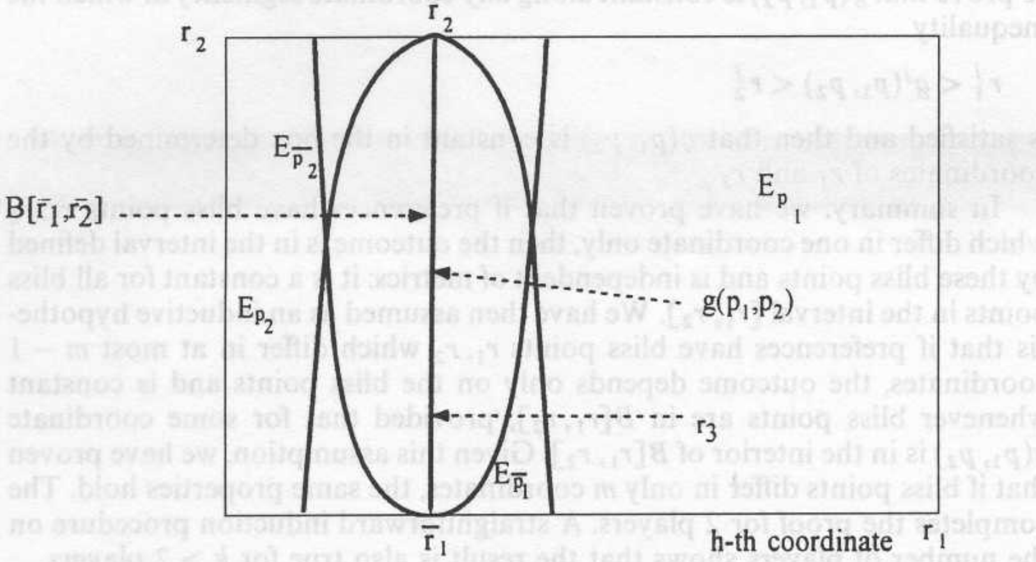


**Fig. 13.** $g$ is constant for all $p_1$ and $p_2$ whose bliss points are in a box determined by $r_1$ and $r_2$

the condition

$$\bar{r}_1^j < g^j(p_1, \bar{p}_2) < \bar{r}_2^j$$

is still satisfied.

By (1) of the inductive hypothesis this implies $g(\bar{p}_1, \bar{p}_2)$ is independent of the metrics for all $p_1, p_2$ whose bliss points are in the $n-(m-1)$ dimensional box determined by $\bar{r}_1$ and $\bar{r}_2$. By (2) of the inductive hypothesis $g$ is a constant for all preferences whose bliss points are in the $n-(m-1)$ dimensional box determined by $\bar{r}_1$ and $\bar{r}_2$. This will be shown to imply immediately that $g(p_1, p_2)$ is a constant in the box determined by $r_1$ and $r_2$; in particular the map $g$ is independent of the metric in this case. Fig. 13 below illustrates the argument:

Consider $\bar{p}_1$ and $\bar{p}_2$ such that their bliss points $\bar{r}_1, \bar{r}_2$ have one more coordinate in common than do the bliss points $r_1, r_2$ of $p_1$ and $p_2$. Let this common coordinate be the $h$-th coordinate, and let this coordinate be the $h$-th coordinate of $g(p_1, p_2)$. Condition (1) of the inductive hypothesis is satisfied, so that $g$ is independent of the metric if bliss points are in $B[\bar{r}_1, \bar{r}_2]$.

Since $g$ is straightforward, the $h$-th coordinate of the outcome $g(\bar{p}_1, \bar{p}_2)$ is in the interior of the segment $(\bar{r}_1^h, \bar{r}_2^h)$ by the following arguments (which are an extension of those illustrated in Fig. 11 and used in the case of preferences whose bliss points differ in one dimension only at the start of step 3 of this proof, see Fig. 13 for an illustration). Suppose that $g(\overline{p_1}, \overline{p_2}) \notin B[\overline{r_1}, \overline{r_2}]$. Then consider the point in $B[\overline{r_1}, \overline{r_2}]$ nearest to $g(\overline{p_1}, \overline{p_2})$: call this point $\overline{r_3}$. Now we can generalize the one-dimensional argument at the start of step 3 of the proof which is illustrated in Fig. 13. Hence $g(\overline{p_1}, \overline{p_2}) \in \textit{interior } B[\overline{r_1}, \overline{r_2}]$, and of course the outcome is independent of the metrics announced.

Now we can use (2) of the inductive assumption to assert that $g$ is also constant within the $n-(m-1)$-dimensional box determined by $\bar{r}_1$ and $\bar{r}_2$. In figure 13 this implies $g(p_1, p_2)$ is constant for $p_1, p_2$ whose bliss points are in $B[\overline{r_1}, \overline{r_2}]$, as in this case $g(p_1, p_2) \in B[\overline{r_1}, \overline{r_2}]$. A similar argument can be given to prove that $g(p_1, p_2)$ is constant along *any* coordinate segment $j$ in which the inequality

$$r_1^j < g^j(p_1, p_2) < r_2^j$$

is satisfied and then that $g(p_1, p_2)$ is constant in the box determined by the coordinates of $r_1$ and $r_2$.

In summary: we have proven that if preferences have bliss points $r_1, r_2$ which differ in one coordinate only, then the outcome is in the interval defined by these bliss points and is independent of metrics: it is a constant for all bliss points in the interval $[r_1, r_2]$. We have then assumed as an inductive hypothesis that if preferences have bliss points $r_1, r_2$ which differ in at most $m-1$ coordinates, the outcome depends only on the bliss points and is constant whenever bliss points are in $B[r_1, r_2]$, provided that for some coordinate $g(p_1, p_2)$ is in the interior of $B[r_1, r_2]$. Given this assumption, we have proven that if bliss points differ in only $m$ coordinates, the same properties hold. The completes the proof for 2 players. A straightforward induction procedure on the number of players shows that the result is also true for $k > 2$ players.

We therefore know that if $g$ is straightforward, $g$ is independent of the metric announced and depends only on bliss points. By the proof in step 2 of the first part of the case $n > 1$, this implies that $g$ is separable on $A^k$, thus implying that $g$ is locally constant or dictatorial on $A^k$.

Consider finally a set of strategies $(p_1, \ldots, p_k) \in (R^n)^k - A^k$. Assume first that a subset of preferences, say $p_1, \ldots, p_j, j < k$ does not belong to $A$. Then

$$g(p_1, \ldots, p_k) = g(\pi(p_1), \ldots, \pi(p_j), \ldots, p_k),$$

where $\pi(p_i)$ is the preference with same metric as $p_i$ and bliss point in the intersection of $\partial A$ and the half line $(r_i, g(p_1, \ldots, p_k)]$. Since, by construction $(\pi(p_1), \ldots, \pi(p_j), \ldots,) \in A^k$ then it follows by the first part of this proof that $g(\pi(p_1), \ldots, \pi(p_j), \ldots, p_k)$ does not depend on the metric of the preferences $\pi(p_1), \ldots, (p_j), \ldots, p_k$. Hence neither does $g(p_1, \ldots, p_k)$ depend on these metrics. Therefore the map $g$ is independent of the metrics in this case also. So we have shown that if $g$ is straightforward then $g$ is separable, by step 2 of this proof.

Now, since for all $(p_1, \ldots, p_k) \in (R^n)^k$

$$g(p_1, \ldots, p_k) = g(\pi(p_1), \ldots, \pi(p_j), \ldots, p_k),$$

where $\pi(p_i) \in \partial A$ by construction, then, for all $i$, the manipulation set $M_{P-i}$ corresponding to any profile $(p_1, \ldots, p_k)$ in $(R^n)^k$ is an affine subspace: this is because the first part of this proof for profiles in $A^k$ now applies. The rests of the argument then follows: since for each $i$, $g(p_1, \ldots, p_{-i})$ minimizes the distance between $r_i$ and $M_{P-i}$ (where $r_i$ is the bliss point of $p_i$), it follows that the map $g(p_1, \ldots, p_k) = g(r_1, \ldots, r_k)$ is separable in this one as well. The first part of the proof of the theorem (for $n = 1$) is now applied and it proves that $g$ straightforward implies $g$ is LCD. The converse follows from Lemma 5. This completes the proof of the characterization theorem. ■

### A.2. Proofs of non-robustness

A topological space $X$ is *second countable* if it has a countable base of neighborhoods for its topology.

A continous map $f: X \to Y$, $X$ and $Y$ topological spaces is called *open* if the image of any open set $U$ in $X$, $f(U)$, is open in $Y$. Note that open maps have the property that if $D$ is a dense subset of $Y$, then $f^{-1}(D)$ is dense in $X$.

A set is *residual* if it is the intersection of (at most) countably many open dense sets. The Baire Category theorem asserts that a residual subset of a complete metric space is dense.

Let $H = \{f: I^k \to \Re, f \text{ a bounded } C^{k+1} \text{ map}\}$. $H$ is a linear space, with the addition rule $(f+g)(x) = f(x) + g(x)$. The $C^{k+1}$ sup norm $\|.\|_{k+1}$ on $H$ is defined by

$$\|f-g\|_{k+1} = \sup_{x \in I^k} \left( \sum_{j=0}^{k+1} \|D^j f(x) - D^j g(x)\| \right).$$

where $D^0(f)$ denotes $f$.

Endowed with the $C^{k+1}$ norm, $H$ is a Banach space, and in particular, a complete metric space.

**Theorem 3.** *The set of non-straightforward games on a bounded choice space is a residual set of the space of all continuous maps from $I^{n,k}$ to $I^n$, $C^0(I^{n,k}, I^n)$, and in particular is a dense set.*

*Proof.* The strategy of the proof is as follows. Let $L$ be the set of continuous maps which are locally constant or dictatorial. We shall consider first the two simplest cases: when a rule $g$ in $L$ is *dictatorial*, and when it is *constant*. We prove that for any dictatorial rule $g$ and dictator $d$, and any small $\varepsilon > 0$, there exists a map $g_\varepsilon$ in the complement of $L$, $C(L) \subset C^0(I^{n,k}, I^n)$, with $\|g_\varepsilon - g\|_{\sup} < \varepsilon$. The proof will then be extended to include rules which are only locally dictatorial, or locally constant, thus implying that $C(L)$ is dense in $C^0(I^{n,k}, I^n)$. Finally, we shall prove that $C(L)$ is open.

Let $C(\nabla) = \{p \in I^k: p = (p_1, \ldots, p_k), p_i \neq p_j \text{ for } i \neq j\}$.

Let $g: I^{n,k} \to I^n$ be a dictatorial map, with dictator $d$. For any $\varepsilon > 0$, $\varepsilon < 1/2$, let $f$ be a $C^1$ diffeomorphism $f: I^n \to \mathrm{I}^n$, such that

$$\sup_{b \in I} \|(f(b) - b)\| < \varepsilon,\ Df(b) \neq 0 \text{ for all } b \in I^n$$

and (7)

$$f(b_d) \neq b_d \quad \text{for some } b_d \in I^n.$$

Consider now the composition map

$$g_\varepsilon = f \circ g: I^{n,k} \to I^n.$$

Then, by construction $\|g_\varepsilon - g\|_{\sup} < \varepsilon$. Now, since for all $p \in I^{n,k}$, $Df(p) \neq 0$ and $Dg(p) \neq 0$, it follows that $Dg_\varepsilon(p) \neq 0$ for all $p$. Therefore $g_\varepsilon$ is nowhere locally constant.

Consider now $p \in g^{-1}(b_d) \subset I^{n,k}$, $p \in C(\nabla)$. Then $g_\varepsilon = f \circ g(p) = f(b_d) \neq b_d$ by construction. Therefore $g_\varepsilon$ is not dictatorial because if it were dictatorial then $g(p_1, \ldots, p_k) = p_d$ for some $d$, whatever $p_d$, and $g_\varepsilon \neq p_d$ for some $b_d \in I$.

Finally, $g_\varepsilon$ is not locally dictatorial (with dictator other than $d$) at $p$, because by construction

$$\frac{\partial g_\varepsilon}{\partial p_j}(p) = Df(p) \cdot \frac{\partial g}{\partial p_j}$$

and $\partial g/\partial p_j = 0$ if $j \neq d$. Note that for all $j$, $\partial g/\partial p_j$ exists because $g$ is dictatorial and, in particular, differentiable. Therefore $g_\varepsilon$ is in $C(L)$. Since $\varepsilon$ is arbitrarily chosen, any dictatorial rule $g$ is a limit of rules in $C(L)$.

Now, consider any constant rule $g: I^{n,k} \to I^n$, $g(p_1, \ldots, p_k) = b_0 \in I^n$, $b_0$ a constant. By the Stone Weierstrass theorem, for any $\varepsilon > 0$ there exists a $C^1$ map $g_\varepsilon$ such that

$$\| g_\varepsilon - g \|_{\sup} < \varepsilon \tag{8}$$

because $I^{n,k}$ is compact. We can obviously require, furthermore, that $Dg_\varepsilon(p) \neq 0$ for $p \in C(\nabla)$. Therefore $g_\varepsilon$ is not locally constant at $p$.

Note that such $g_\varepsilon$ cannot be locally dictatorial at $p$ either because it is in an $\varepsilon$-neighborhood of the constant map $g$, and $\varepsilon$ is arbitrarily chosen: any locally dictatorial map on a set $U \subset C(\nabla)$ will be at least at a positive distance $\varepsilon_0$ from the constant rule $g$, $\varepsilon_0$ a constant depending on the set $U$ and on $b_0$.

Since $\varepsilon$ is arbitrarily chosen, any constant rule $g$ is a limit of rules in $C(L)$.

Consider now an arbitrary straightforward $g$ in $L$, and let $p \in C(\nabla)$. Then there exists a neighborhood $U$ of $p$ such that either $g/_U$ is dictatorial, or $g/_U$ is a constant map.

The argument given above for constant and dictatorial maps, when restricted to $U \subset P^k$, prove that $g/_U$ can be arbitrarily approximated by a rule $g_\varepsilon(U)$ defined on $U$, and such that $g_\varepsilon(U)$ is neither locally constant nor locally dictatorial on $U$. A standard argument using partitions of unity (see, e.g. Guillemin and Pollak [14, p. 52]) can then be used to prove the existence of a continuous map $g_\varepsilon: I^{n,k} \to I^n$ such that $g_\varepsilon/_U = g_\varepsilon(U)$ and $\| g_\varepsilon - g \|_{\sup} < \varepsilon$.

Since $g_\varepsilon(U)$ is not locally constant nor locally dictatorial at $p$, $g_\varepsilon$ is not either. Thus $g_\varepsilon$ is a function in $C(L)$ within an $\varepsilon$-neighborhood of $g$. Therefore $C(L)$ is dense in $C^0(I^{n,k}, I^n)$.

Next we prove that it is open. Consider now $g \in C(L)$. Let $p \in C(\nabla)$ be such that $g$ is neither locally constant, nor locally dictatorial at $p$.

The fact that $g$ is not locally dictatorial at profile $p$ implies that in any neighborhood $U_p$ of $p$, there exists for each $j = 1, \ldots, k$ a profile $p^j$ and a number $\varepsilon^j > 0$ such that

$$|g(p^j) - g_j(p^j)| = \varepsilon^j,$$

where $b_j(p^j)$ is the bliss point of the $j$-th. preference in the profile $p^j$. Now for any $\rho \in C^0(I^{n,k}, I^n)$,

$$|\rho(p^j) - b_j(p^j)| \geq \varepsilon^j - \| \rho - g \|_{\sup}.$$

From this inequality it follows that no rule within an $\varepsilon^j/2$ neighborhood of $g$ in $C^0(I^{n,k}, I^n)$ can be locally dictatorial with dictator $j$ at $p$. This is because any such rule will satisfy

$$|\rho(p^j) - b_j(p^j)| \geq \varepsilon^j/2$$

at any profile $p^j$ where $g$ satisfies

$$|\phi(p^j) - b_j(p^j)| = \varepsilon^j,$$

and every neighborhood of $p$ will contain such points. If we now set $\varepsilon = \min_{j=1,\ldots,k} \varepsilon^j$, then no rule $\rho$ within an $\varepsilon/2$ neighborhood of $g$ in $C^0(I^{n,k}, I^n)$ is locally dictatorial at $p$.

As above, let $p \in C(\nabla)$, $g \in C(L)$ not locally constant at $p$, and let $q \in U_p$ be such that

$$|g(p) - g(q)| > \varepsilon^1$$

for some $\varepsilon^1 > 0$; such a $q$ exists because $g$ is not locally constant. Then

$$|\rho(p) - \rho(q)| > \varepsilon^1 - 2\|\rho - g\|_{\sup}.$$

Therefore for $\delta = \frac{1}{2}\min(\varepsilon, \varepsilon^1)$, any rule $\rho$ within a $\delta$ neighborhood of $g$ in the sup topology is neither locally constant nor locally dictatorial at $p$. Therefore $C(L)$ is open, completing the proof. ■

A.2.1. Results on Nash equilibria

**Theorem 4.** *Let $\phi$ be a continuous social choice rule, $\phi: P^k \to A$, where $P = A = I$, the unit interval in $\Re$, and $k \geq 2$. Let $M = I$ be the message space consisting of statements on bliss points of individual preferences. If the rule $\phi$ is Nash implementable by a regular game $g: M^k \to A$, then $\phi$ is locally constant or dictatorial (LCD).*

*Proof.* Consider the reaction set $R_{p_i}$ corresponding to individual $i$ with preference $p_i \in P$, i.e., the set of message-profiles

$$R_{p_i} = \{(m_i(m_{-i}), m_{-i}) \in M^k \colon (m_i(m_{-i}), m_{-i}) \in \arg\max_{m_i \in M}(p_i(m_i, m_{-i}))\}.$$

This is the set of vectors $m \in M^k$ such that agent $i$'s message is his or her optimal response to the messages of the other agents, i.e., to $m_{-i}$. It is a generalization of the concept of reaction function. Figure 14 illustrates this for two agents: the level sets of a game $g: \Re^2 \to \Re^1$ are shown, and agent 1's reaction set is the locus of points of tangency between these curves and horizontal lines corresponding to strategy choices by agent 2.

The strategy of the proof is as follows.

1. In step one we show that if preferences are such that a Nash equilibrium message profile in $M^k$ is one at which no agent attains her bliss point, i.e., $b_i \neq g(\overline{m}) \forall i$, then the social choice rule is locally constant in a neighborhood of these preferences. This follows from two facts: that $b_i \neq g(\overline{m}) \forall i$ is an open condition, and that the profile of agent's messages at a Nash equilibrium satisfies simultaneous optimality conditions as each is a best response to the others. We show that these same optimality conditions continue to characterize agents' best responses for small changes in preferences, and that in this case the outcome of the social choice rule must be locally independent of the agents' preferences.
2. In step two we consider the case in which a Nash equilibrium gives as an outcome the bliss point of one agent, and show that in this case the social choice rule is locality dictatorial with that agents as dictator.

*Step 1.* No agent attains her bliss point at a Nash equilibrium

Now consider the set $T_{p_i} = R_{p_i} - g^{-1}(b_i)$, which is agent $i$'s reaction set minus the preimage of her bliss point.[21] Let $\Pi_{-i}$ be the projection of a vector

[21] Obviously, if $b_i \notin g(M^k)$ then $T_{p_i} = R_{p_i}$.

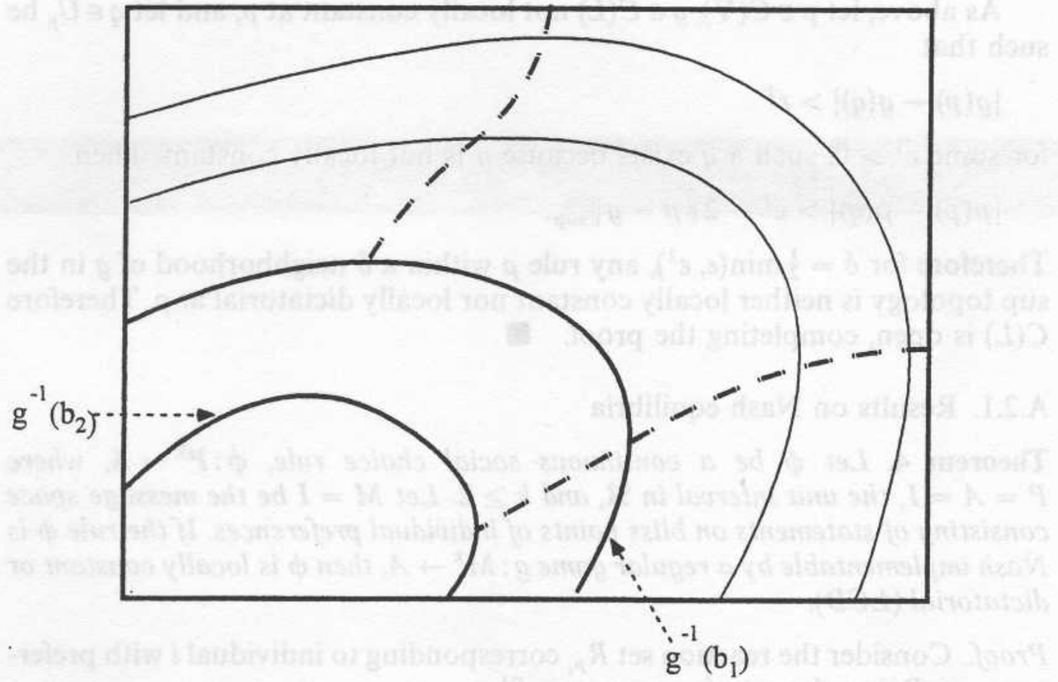


**Fig. 14.** The sets $g^{-1}(b_i)$, $i = 1, 2$

in $M^k$ onto all coordinates other than the $i$-th. Note that for any set of strategies of players other than $i$, $m_{-i} \in M^{k-1}$, either this is in the projection onto coordinates other than the $i$-th of the preimage of $i$'s bliss point, i.e., $m_{-i} \in \Pi_{-i}(g^{-1}(b_i))$ or it is not, i.e., $m_{-i} \in \Pi_{-i}(T_{p_i})$. In the first case, agent $i$'s best response is obvious: it is $m_i$. Consider next the second case and let $m_{-i} \in \Pi_{-i}(T_{p_i})$. Since by definition of $T_{p_i}$, the $i$-th. individual is unable to obtain $b_i$ as an outcome of the game in response to $m_{-i}$, $i$'s preference-maximizing response is a message that minimizes the difference

$$g(m_i, m_{-i}) - b_i,$$

by definition of the preferences $p_i$ in $P$.

For all $m_{-i} \in M^{k-1}$, the manipulation set $\mathscr{M}_{m-1}$ [22] is a connected set in $\Re$: this is an application of the mean value theorem to the continuous real valued function of one variable $g(., m_{-i}): I \to \Re$. Therefore, since $(m_i, m_{-i}) \in g^{-1}(b_i)$, the problem

$$\min_{m_i \in M} (g(m_i, m_{-i}) - b_i)$$

is necessarily equivalent to one of the following optimization problems:

a) $\max_{m_i \in M} g(m_i, m_{-i})$ if $\sup \mathscr{M}_{m_{-i}} < b_i$,

[22] We are using a script letter, $\mathscr{M}$, to denote the manipulation set in this section to avoid confusion with the message space $M$.

or

b) $\min_{m_i \in M} g(m_i, m_{-i})$ if $b_i < \inf \mathscr{M}_{m_{-i}}$.

Since by assumption $g$ is $C^{k+1}$, any optimal $m \in T_{p_i}$ satisfies first order orthogonality condition on its gradient $Dg$. If $(m_i, m_{-i})$ is in the interior of $M^k$, $\overset{\circ}{M}{}^k$, then

$$Dg \perp N_{m-i}, \tag{9}$$

where $N_{m-i} = \{(m_i, m_{-i}) \in S^k: m_i \in M\}$ is the premanipulation set of agent $i$ given the strategies $m_{-i}$ of other agents. It is the intersection of one coordinate axis in $\mathfrak{R}^k$ with $M^k$. Therefore, $Dg \perp N_{m-i}$ implies $Dg_i = 0$ if $m = (m, m_i) \in \overset{\circ}{M}{}^k$. If instead, $m$ is in the boundary of $M^k$, $\partial M^k$, then $m$ belongs to a face $F_\beta$, i.e. a subset of $M^k$ characterized by having all coordinates except for those in the set $\beta \subset \{1, \ldots, k\}$ constant, and equal either to zero or to one. In this case, the orthogonality condition for optimality is

$$\Pi_\beta(Dg) \perp N_{m-i}, \tag{10}$$

where $\Pi_\beta$ denotes the projection map from $\mathfrak{R}^k$ into $\mathfrak{R}^\beta$, the Euclidean space with coordinates in $\beta$. By definition, this latter orthogonality condition implies that the $i$-th. coordinate of $\Pi_\beta(Dg)$ must vanish, i.e. that $Dg_i = 0$ if $i \in \beta$. Note that in addition to (9) and (10), the solutions to problem (a) and (b) must satisfy second order conditions, and must be global. Let $g(., m_{-i}): I \to I$, and $\bar{m} = (\bar{m}_i, \bar{m}_{-i})$ be in $T_{p_i}$. Then in the case of (a) $\partial^2 g(\bar{m})/\partial m_i^2 \le 0$ for all $i$. Now taking $\eta = \{i\}$, the regularity condition (1) implies $g_i \pitchfork \Delta_i$. Note that $(\bar{m}, g(\bar{m})) \in g_i^{-1}(\Delta_i)$ since $g_i(\bar{m}, g(\bar{m})) = (Dg_i(\bar{m}) + g(\bar{m}), g(\bar{m})) = (g(\bar{m}), g(\bar{m}))$. Therefore, by (1) $g_i(\bar{m}, f(\bar{m})) \pitchfork \Delta_i$, i.e., $D(Dg_i(\bar{m}) + g(\bar{m})) \neq Dg(\bar{m})$, implying $\partial^2 g(\bar{m})/\partial m_i^2 \neq 0$. Therefore, in the case of (a) if $\bar{m} \in T_{p_i}$

$$\frac{\partial^2 g(\bar{m})}{\partial m_i^2} < 0. \tag{11}$$

Similarly, in the case of (b)

$$\frac{\partial^2 g(\bar{m})}{\partial m_i^2} > 0 \text{ if } \bar{m} \in T_{p_i}. \tag{12}$$

Consider now a profile $(\bar{p}_i, \ldots, \bar{p}_k)$ in $P^k$ and let $\phi(\bar{p}_i, \ldots, \bar{p}_k) = \bar{c}$ in $A$. Since $g$ Nash-implements $\phi$ by assumption, there exists at least one message profile denoted $\bar{m} = \bar{m}(\bar{p}_i, \ldots, \bar{p}_k) \in M^k$, which is a Nash equilibrium of the game form $g$ with preferences over outcomes $(\bar{p}_i, \ldots, \bar{p}_k)$, satisfying

$$g(\bar{m}) = \phi(\bar{p}_i, \ldots, \bar{p}_k).$$

Now, by definition, $\bar{m} \in \cap_{i=1}^k R_{\bar{p}_i}$. Recall that, for each $i$, $R_{\bar{p}_i} = T_{p_i} \cup g^{-1}(\bar{b}_i)$, where $\bar{b}_i = \bar{b}_i(\bar{p}_i)$ is the bliss point of $\bar{p}_i$.

We shall consider first the case in which $\overline{m} \notin g^{-1}(\bar{b}_i)$, and show that $\bar{m}$ is a Nash equilibrium for any preference profile $p$ in some neighborhood $V$ of $\bar{p}$ in $P^k$. Since $g$ Nash implements $\phi$, this will imply that for all $p = (\bar{p}_i, \ldots, \bar{p}_k)$ in $V$,

$$\phi(\bar{p}_i, \ldots, \bar{p}_k) = g(\bar{m}) = \bar{c}$$

i.e. $\phi$ is a constant on $V$.

Observe that the orthogonality conditions (9) and (10) (valid for $M^k$ and $\partial M^k$ respectively) are only dependent on the gradient of the game form $Dg$ at $\bar{m}$, and not on the chosen profile $\bar{p}$. With respect to the second order conditions (11) and (12) (associated with problems (a) and (b) respectively) these will be satisfied in some neighborhood $w$ of $\bar{p}$ in $P^k$ when they are satisfied at $\bar{p}$, since they are open conditions: small variations of the preference profiles $\bar{p}$ in $W$ are associated with small variations of the corresponding bliss points $(\bar{b}_1, \dots, \bar{b}_i)$, so that if for some $j$ and $m_{\bar{j}}$, $\sup(F(m_{\bar{j}})) < \bar{b}_j$, then $\sup(F(m_{\bar{j}})) < b_j)$, for $b_j = b_j(p_j)$, and $p_j$ in a $\varepsilon$-neighborhood $N_\varepsilon$ of $\bar{p}_j$, and if for some $i$ and $m_{\bar{i}}$ $\inf(F(m_{\bar{i}})) > \bar{b}_i$, then $\inf(F(m_{\bar{j}})) > b_i$ for $b_i = b_i(p_i)$, $p_i$ in an $\varepsilon$-neighborhood $N_\varepsilon$ of $\bar{p}_i$. Note that the $\varepsilon$'s of $N_\varepsilon$ can be chosen uniformly (for all $m_{\bar{j}}$ in $M^{k-1}$) because of the compactness of $M^{k-1}$.

Hence $\bar{m}$ satisfies both first and second order conditions for all profiles $p$ in a neighborhood $W$ of $\bar{p}$. In addition, the components of $\bar{m}$ will be globally optimal responses for all profiles near enough to $p$.[23] It follows that $\bar{m}$ is a Nash equilibrium of the game form $g$ for all profiles of preferences $p$ in some neighborhood $V$ of $\bar{p}$, $V \subset W$. Thus $\phi$ is locally constant at $\bar{p}$ in this case.[24] This completes step one of the proof.

**Step 2.** i's bliss point is a Nash equilibrium.

Consider now the case in which the Nash-equilibrium set of messages $\bar{m}$ associated with a profile $\bar{p} = (\bar{p}_1, \dots, \bar{p}_k)$ is in $g^{-1}(\bar{b}_d)$, for some $d = 1, \dots, k$. We shall show that in this case the social choice rule $\phi$ is locally dictatorial. Note that if $\bar{m} \in g^{-1}(\bar{b}_i) \cap g^{-1}(\bar{b}_j)$, then $\bar{b}_i = \bar{b}_j$, since the hypersurfaces of a function do not intersect.

We now show that if for all $i, j = 1, \dots, k$, $\bar{b}_i \neq \bar{b}_j$, then there exists a neighborhood $U$ of $\bar{p}$ in $P^k$ such that the Nash equilibria corresponding to any profile $p$ in $U(\bar{p})$, $\bar{m}$, are also in $g^{-1}(b_d)$, where $b_d = b_d(p_d)$ is the bliss point of the $d$-th. preference $p_d$ in $p$. Since $g$ Nash-implements $\phi$, this implies that $\forall p$ in $U(\bar{p})$, $\phi(\bar{p}_1, \dots, \bar{p}_k) = g(\bar{m}(p)) = b_d$, i.e. $\phi$ is locally dictatorial at $\bar{p}$, with dictator $d$. This would complete the proof that $\phi$ is LCD when it is Nash-implementable by a regular game.

Since $\overline{m}(\bar{p}) \in g^{-1}(\bar{b}_d)$ by assumption, if $\overline{m} \in \mathring{M}^k$ it follows that for $\eta = \{1, \dots, d-1, \dots, k\}$ (the set of integers from 1 to $k$ with $d$ deleted) the couple $(\overline{m}, \bar{b}_d) \in g_\eta^{-1}(\Delta_\eta)$ i.e.,

$$(Dg_{\eta_1}(\bar{m}) + g(\bar{m}), \dots, Dg_{\eta_{k-1}}(m) + g(\bar{m}), g(\bar{m})) = (\bar{b}_d, \dots, \bar{b}_d) \in \Delta_\eta,$$

since $g(\bar{m}) = \bar{b}_d$, and for all agents $j \neq d$ messages will be chosen to satisfy orthogonality conditions. By the regularity assumption (1) $g_\eta \pitchfork \Delta_\eta$, and $g_\mu \pitchfork \Delta_\mu$ for any $\mu \subset \eta$, implying that the map $\partial g_\eta$, the restriction of $g_\eta$ on $\partial(M^k \times I)$, satisfies $g_\mu \pitchfork \Delta_\mu$.

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[23] Suppose $\sup(F(\bar{m}_{\bar{j}})) < \bar{b}_j$. Then as noted above $\sup(F(\bar{m}_{\bar{j}})) < b_j$ for $b_j(p_j)$, $p_j$ in $N_\varepsilon$ of $\bar{p}_j$. Hence if $\bar{m}_j$ solves the problem $\max_{m_j \in M} g(m_j, \bar{m}_{\bar{j}})$ globally, then it is the globally optimal response for any $p_j \in \bar{p}_j$.

[24] One can actually show that if $\bar{m} \in \cap_{i=1,\dots,k} T_{\bar{p}_i} \forall \bar{p}_i \in p$, then $\exists$ a neighborhood $N$ of $\bar{p}$ such that $m \in \cap_{i=1,\dots,k} T_{p_i} \forall p_i \in p$ in $N(\bar{p})$. Since $\bar{m} \in \cap_{i=1,\dots,k} T_{\bar{p}_i}$, then by definition $g(\bar{m}) \neq \bar{b}_i \forall \bar{b}_i$ corresponding to the profile $\bar{p}_i$, i.e., $\bar{m} \notin \cup_{j=1,\dots,k} g^{-1}(b_j)$ for $b_j$ sufficiently close to $\bar{b}_j$. Thus $\bar{m} \in R_{p_i} - g^{-1}(b_i) \forall p_i \in p \in N(\bar{p})$, thus implying by definition that for all $p \in N$, $\bar{m} \in \cap_{i=1,\dots,k} T_{p_i}$, $p_i \in p$.

Now, $g_\eta : M^k \times I \to \Re^k$, and $\Delta_\eta$ is a one dimensional submanifold of $\Re^k$. Therefore, by the transversality theorem [14, 1974, p. 60], $g_\eta^{-1}(\Delta_\eta)$ is a one dimensional submanifold of $M^k \times I$ (possibly with boundaries and corners). Therefore, there exists a neighborhood $U$ of $(\bar{m}, \bar{b}_d)$ in $M^k \times I$ and a $C^1$ curve $b_d \to (m(b_d), b_d)$, for all $b_d \in \Pi_{k+1}(U)$, the projection of $U$ onto its $k+1$-th. coordinate, such that $(m(b_d), b_d)$ is contained in $g_\eta^{-1}(\Delta_\eta)$, i.e.

$$g(m(b_d)) = b_d \text{ and } Dg_{\eta_1}(m(b_d)) = \ldots = Dg_{\eta_{k-1}}(m(b_d)) = 0.$$

We shall now show that $U$ can be chosen sufficiently small that the $C^1$ curve $m(b_d)$ in $M^k$ consists of Nash equilibria corresponding to preference profiles $p = (p_1, \ldots, p_k)$ in some neighborhood of $\bar{p}$ in $P^k$.

We know that for all $j$ in $\eta$, $Dg_j(m(b_d)) = 0$ because $(m(b_d), b_d) \in g_\eta^{-1}(\Delta_\eta)$ by construction. Therefore all message profiles in the curve $m(b_d) \subset \Pi_{-(k+1)}(U) \subset M^k$ satisfy the first order conditions (1) (which are independent of the preference profiles). Recall that $\Pi_{-(k+1)}$ is the projection map on all coordinates but $k+1$.

Consider now a profile $\tilde{p}$ of the form $(\bar{p}_1, \ldots, \tilde{p}_d, \ldots, \bar{p}_k)$ where all but the $d$-th. preference are as in the profile $\bar{p}$, and such that the bliss point $b_d$ corresponding to $\tilde{p}_d$ is in $\Pi_{k+1}(U)$, the projection of $U$ onto its $k+1$-th. coordinate. Then when $U$ is sufficiently small, $m(\bar{b}_d)$ is a Nash equilibrium for $(\bar{p}_1, \ldots, \tilde{p}_d, \ldots, \bar{p}_k)$. To see this, note first that $g(m(\tilde{b}_d)) = \tilde{b}_d$, so that the $d$-th. individual strategy is indeed optimal. Next note that $m(\tilde{b}_d)$ satisfies the first order conditions (9), (10) as shown above, and the second order conditions (11) and (12) corresponding to problems (a) and (b) by the openness of these conditions. By continuity and an argument similar to step one and that in the last footnote above, $U$ can be chosen small enough that $m_j(\tilde{b}_d)$ is globally optimal for $j \neq d$. This proves the point.

We have therefore shown that any message $m(\tilde{b}_d)$ in the curve $m(b_d) \subset \Pi_{-(k+1)}(U)$ is a Nash equilibrium for a preference profile $\bar{p} = (\bar{p}_1, \ldots, \tilde{p}_d, \ldots, \bar{p}_k)$ in a neighborhood $U$ of $\bar{p} \in p^k$, where $\tilde{b}_d$ is the bliss point of $\tilde{p}_d$. Furthermore, we have also shown that any message in the curve $m(b_d) \subset \Pi_{-(k+1)}(U)$ is in the set $T_{\bar{p}_j}$ for all $j$ within the set of indices $\eta$, where $\bar{p}_j$ is the $j$-th. preference in the profile $(\bar{p}_1, \ldots, \bar{p}_k)$.

This implies that to each preference profile of the form $p = (\bar{p}_1, \ldots, \tilde{p}_d, \ldots, \bar{p}_k)$ in a neighborhood $U$ of $\bar{p}$, corresponds to a Nash equilibrium in the curve $m(\tilde{b}_d) \subset \Pi_{-(k+1)}(U)$, namely $m(\tilde{b}_d)$. Note that there could be Nash equilibria other than $m(\tilde{b}_d)$ associated to $\bar{p}$. However, in order to know the value of $\phi$ at $\bar{p}$ it suffices to know that $m(\tilde{b}_d)$ is a Nash equilibrium of $\bar{p}$: as $g$ Nash implements $g$,

$$\phi(\tilde{p}) = g(m(\tilde{b}_d)) = \tilde{b}_d \quad \text{for all } \bar{p} \text{ in } N \text{ of the form } \bar{p} = (\bar{p}_1, \ldots, \tilde{p}_d, \ldots, \bar{p}_k)$$

We now use the results of step 1 above.

Given that $m(\tilde{b}_d)$ is a Nash equilibrium for $\bar{p}$, and that $d$ is a dictator at $m(\tilde{b}_d)$, it follows that for all $j \neq d$, $m(\tilde{b}_d)$ must be in $T_{\bar{p}_j}$. But as we saw in step 1 this implies that there exists for all $j \neq d$ an $\varepsilon$-neighborhood $N_\varepsilon$ of $\bar{p}_j$ such that $m(\tilde{b}_d)$ is in $R_{p_j}$ for any $p_j$ in $N_\varepsilon$. We have therefore proven that for all profiles $p = (p_1, \ldots, p_d, \ldots, p_k)$ in some neighborhood $W$ of $\bar{p}$ in $P^k$ there exists a Nash equilibrium in $\Pi_{-(k+1)}(U)$, namely $m(b_d)$, where $b_d$ is the bliss point of $p_d$. It follows by construction of the curve $m(b_d)$ that for all $p$ in $W$,

$\phi(p) = g(m(b_d)) = b_d$, so that $\phi$ is dictatorial in $\mathring{W}$, with dictator $d$. This completes the proof of step 2, when $\bar{m} \in \mathring{M}^k$. A similar proof applies for $\bar{m} \in \partial M^k$, e.g. $\bar{m} \in F_\eta \subset M^k$, where $F_\eta$ is face in $M^k$ with all coordinates but those in $\eta$ constant, since condition (1) applies for all $\eta$. This completes the proof of the theorem. ■

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