Soc Choice Welfare (1997) 14: 303–318

Social Choice and Welfare

Social choice with infinite populations: construction of a rule and impossibility results

Graciela Chichilnisky[1], **Geoffrey Heal**[2]

[1] Department of Economics, Columbia University, 405 Law Memorial Library, New York, NY 10027, USA
[2] Program on Information and Resources, Columbia University, 405 Law Memorial Library, New York, NY 10027, USA

Received: 30 October 1994/Accepted: 22 April 1996

Abstract. We provide a simple construction of social choice rules for economies with infinite populations. The rules are continuous, Pareto and non-dictatorial; they are constructed as limits of individual preferences when the limit exists, and otherwise as adequate generalizations. This contrasts with the impossibility results of Arrow (1951) and Chichilnisky (1980), which are valid on economies with finitely many individuals. Our social choice rules are, however, limits of dictatorial rules. This paper was written in 1979.

1. Introduction

Social choice seeks ways to aggregate individual preferences into social preferences. The problem is to find maps, called *social choice rules*, from spaces of individual preferences to spaces of social preferences, which satisfy certain desired conditions or *axioms*. We study the problem in the context of large economies, and show that with infinite populations one can construct maps which assign a social preference to each set of individual preferences and satisfy an attractive set of axioms. The maps are continuous, Pareto (which means that if everybody prefers one choice to another then so does the social

---

G. Chichilnisky is at the Department of Statistics, Columbia University, and G.M. Heal was at the Department of Economics, University of Sussex and Columbia University in 1980, and is currently at Columbia Business School. This research was partly supported by the UNITAR project on technology, distribution and North-South relations at the Center for the Social Sciences of Columbia University. The authors thank J.C. Candeal, J. Drèze, C. Fourgeaud, G. Bergman, M. Hirsch, E. Indurain, D. Sonderman for helpful comments.

preference) and non-dictatorial. We also characterize the properties of these rules[1].

The difficulty in constructing social choice rules resides, of course, in the conditions required of them. Three widely used conditions, or *axioms*, are: continuity, anonymity and respect of unanimity[2]. These axioms are best suited for euclidean spaces of choices such as those typically used in economic theory[3]. Because of its affinity with the rest of economic theory, this approach to social choice has led to the discovery of close links between social choice theory and consumer theory[4], and between social choice and classic problems of resource allocation such as the existence of competitive equilibrium and the non-emptiness of the core[5]. Another set of axioms considered for continuous social choice rules consists of the Pareto principle and non-dictatorship.

While these three axioms are simple and appealing, in many cases they may be impossible to satisfy. For example, earlier work by Chichilnisky has shown that any continuous Pareto rule is "almost dictatorial", in the sense that there is an agent who, by appropriate misrepresentation of preferences can obtain any outcome he or she might wish. However, for economies containing infinitely many individuals, there are social choice rules which are continuous, Pareto and non-dictatorial: in this paper we construct such rules in an intuitive fashion, and describe their properties.

We construct a social choice rule as follows. Suppose that an infinite sequence of individual preferences has a limiting preference. Then we simply select that limit as the social preference. This defines a continuous social choice rule which obviously respects unanimity over all such sequences. In particular, this map can be shown to satisfy the Pareto condition. Furthermore this map is not dictatorial.

One then attempts to extend this map to *all* sequences of preferences, whether or not they have limiting preferences. We show that this extension can be achieved while maintaining important properties: the Pareto property and the non-dictatorship properties remain[6]. Therefore, the social choice problem is resolved for economies with infinitely many individuals.

---

[1] This paper was first written in 1979. In recent years its results have been widely circulated, cited and applied, and yet have not been superceded. The following papers have been written building on and using the results of this one: L. Lauwers [28, 29], Efimov and Koshevoy [22], Lauwers and van Liederkerke [30], Candeal, Indurain and Uriarte [5], Candeal, Chichilnisky and Indurain [4], Chichilnisky [16], Chichilnisky [17].

[2] These axioms were introduced in Chichilnisky [7, 8]. They are different from Arrow's [1] axioms for social choice, which are Pareto, independence of irrelevant alternatives and non-dictatorship, although recent work point to deep relations between the two approaches, Baryshnikov [2], also in this issue.

[3] Instead, Arrow's axioms for social choice have been mostly applied to discrete set of choices, and using combinatorial techniques. The main difference in the axioms is that Chichilnisky required continuity while Arrow requires independence from irrelevant alternatives.

[4] See Chichilnisky and Heal [18].

[5] Chichilnisky [11, 13–15].

[6] It is straightforward to show that such a map is Pareto and non-dictatorial. We show that if the point of accumulation is chosen as the limit of a sequence along a free ultrafilter, then this selection can also be done continuously.

The extension to all sequences of preferences is achieved by choosing as a social preference, a point of accumulation of individual preferences. This point of accumulation can be selected in a continuous way but, in general, the result of this selection depends on the order in which the preferences are listed. Hence a permutation of preferences among individuals will affect the social preference, so that the social choice rule, although Pareto and non-dictatorial, is no longer anonymous. Anonymity is a stronger condition than non-dictatorship, a point discussed below. Related results addressing the issue of anonymity for infinite populations have been obtained by Lauwers (this volume), Lauwers and van Liederkirke [30], Candeal et al. [5], Candeal et al. (this volume), Chichilnisky [16, 17], and Effmov and Koshevoy [22].

Infinite populations are an important feature of our results. Our construction of a non-dictatorial rule depends crucially on there being an infinite number of agents: with only a finite number, this rule selects the preference of the last agent, who is then a dictator. Having an infinite population means that there is no "last agent", so one can safely select the limit preference as a social preference without introducing a dictatorship[7]. We show, however, that the social choice rules which we construct are limits of dictatorial rules. In addition, we show that there is a family of individuals of measure as small as desired, which imposes its wishes on society.

2. Notation and definitions

Let X be the *choice space*, $X = R^n$[8]. A *preference* p on X is defined by giving for each choice x in X a preferred direction, a vector denoted $p(x)$, which can be understood as the gradient field of a utility $u: R^n \to R$ representing the preference[9]. A preference is a continuous map $x \to p(x)$ from choices x into vectors in R^n. Such a map is called a *vector field* on X.

As is usual in social choice theory it is the ordering and not the utility values that matter; therefore, the lengths of the vector fields do not matter; one normalizes the vector fields to obtain vectors of unit length, i.e. $\| p(x) \| = 1$ for all x. We topologize the space R^n as usual, and endow the space of preferences with a topology in which the distance between two preferences p, q is the supremum of the euclidean norm of their difference: $\| p - q \| = \sup_{z \in R^n} \| p(x) - q(x) \|$.

[7] Other results on social choice with infinitely many agents were obtained by Fishburn, Kirman and Sonderman, and are discussed below.
[8] It suffices to consider a space which is *homeomorphic* to R^n. A topological space X is *homeomorphic* to another Y if there exists an one to one onto continuous function $f: X \to Y$ whose inverse is also continuous.
[9] The vector $p(x)$ is intended to be the normal to the indifference surface when the preference is represented by a utility function $u: R^n \to R$, so that in effect $p(x)$ is the gradient vector $Du(x)$; in other words, the preferences considered here are representable by utility functions with continuous derivatives, also called smooth preferences, see Debreu [20] and Chichilnisky [6].

Definition 1. The space of preferences P is a closed, equicontinuous family of locally integrable continuous unit vector fields on X[10].

The space of continuous bounded vector fields on R^n is an infinite dimensional space endowed with the above topology, and its subspace P contains infinite dimensional manifolds[11]. P also contains finite dimensional spaces of preferences, such as the space of all linear preferences, denoted P_L.

The space P *is compact* by Ascoli's theorem[12].

We assume that there are infinitely many individuals indexed by the set of integers N.

Definition 2. A profile of preferences in P is a sequence of preferences, one for each individual: $\{p_i\}_{i=1,2\ldots} = (p_1, p_2, p_3, \ldots)$: $\forall i$, $p_i \in P$. A profile is therefore an element of the product of the space P with itself indexed by the integers, denoted $\prod_{i=1}^{\infty} P_i$.

P_i denotes the space of preferences of the ith individual and thus $P_i = P$ for all i. A profile is denoted $\{p_i\}_{i=1,2\ldots}$ or simply $\{p_i\}$. If there are finitely many individuals, say, $i = 1, \ldots, k$, then the space of profiles is, instead, $\prod_i^k P_i$.

Definition 3. A social choice rule ϕ is a map from profiles in P to social preferences in P. With infinitely many individuals, therefore

sas

$$\phi : \prod_{i=1}^{\infty} P \to P$$

and with k individuals

$$\phi : \prod_{i}^{k} P \to P.$$

Continuity of the social rule ϕ is defined with respect to the topology of the spaces of profiles $\prod_i^k P$ and $\prod_{i=1}^{\infty} P$; the former is endowed naturally with the product topology it inherits from P. A product topology ρ for the spaces $\prod_{i=1}^{\infty} P$ will be defined below.

---

[10] I.e. $p(x) \in P \Rightarrow p(x)$ is locally the gradient of a C^1 real valued function on X, see also Chichilnisky [6] and Debreu [20]. The family P of maps from X into R^n is called *equicontinuous* at $x \in X$ iff there is a neighborhood of x whose image is small under every member of P; formally, P is equicontinuous at x when for every neighborhood V of R^n there is a neighborhood $U(x)$ of X such that $\forall p \in P$, $p(U) \subset V(p(x))$. P is eqicontinuous when it is equicontinous at every x. For P to be equicontinuous it suffices that the vector fields in P are continuously differentiable, with uniformly bounded derivatives.

[11] See Chichilnisky [6]. The space P of all locally integrable preferences is characterized by the Frobenius integrability conditions, which are given by a set of partial differential equations: they are necessary (but not sufficient) conditions for a vector field to be the gradient of a real valued function. Under monotonicity conditions, the conditions are also global, see, for instance, Debreu [10]. The Frobenius conditions and the normalization assumption $\| p(x) \| = 1$ are both closed conditions, so that the space P endowed with the C^1 topology is a complete space since it is a closed subspace of the space of all C^1 vector fields.

[12] See Kelley [27, p. 234].

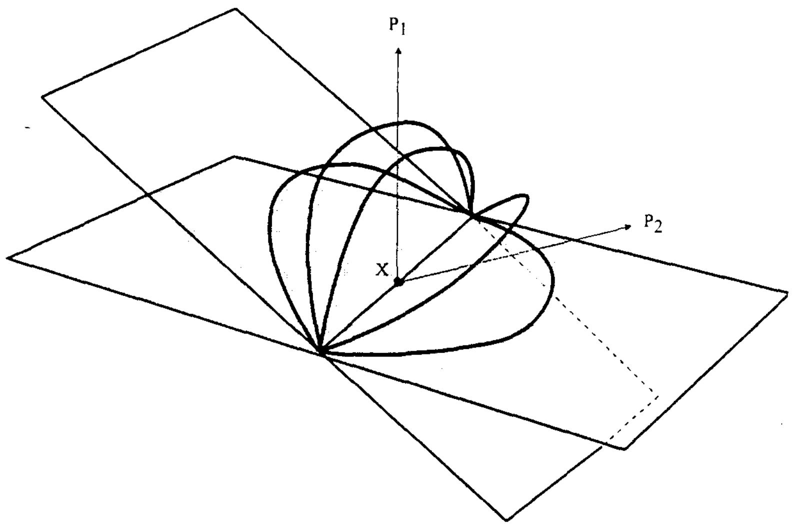

Fig. 1. The Pareto condition for a profile with two different preferences

Definition 4. A social choice rule ϕ is said to satisfy the *Pareto* condition, or simply to be Pareto, if whenever all individuals in the profile $\{p_i\} \in \prod_i P$ prefer choice x to choice y[13] then so does the outcome or social preference assigned to $\{p_i\}$, $\phi(\{p_i\})$.

A simple geometric representation of the Pareto condition is in Lemma 1 below. To motivate it, consider the following example.

Example 5. Fig. 1 represents the case where the choice space is R^3 and there are only 2 individuals with preferences p_1, p_2, defining the profile $\{p_i\}$. If the map $\phi: \prod_i^2 P \to P$ is Pareto then the outcome preference p at a fixed choice x must determine a vector $p(x)$ in the convex cone of vectors having positive inner products with all vectors in the area designated by semi-circles. This is because the Pareto property implies that any utility function representing the outcome preference p at the choice x must increase in the directions that u_i increases, where u_i is a utility representation for the preference p_i, $i = 1, 2$. Therefore $p(x)$ must have positive inner product with all vectors in the shaded area of S^2. These latter vectors are all those that have positive inner product with both p_1 and p_2.

More generally:

Lemma 6. *The map* $\phi: \prod_i P \to P$ *is Pareto if and only if* $\phi(\{p_i\})(x) \in$ *convex hull* $\{p_i(x)\}_{i=1,2\ldots}$[14].

[13] For instance, if the utility representations for the preferences $\{p_i\}_{i=1,2\ldots}$, say $\{u_i\}_{i=1,2\ldots}$, all give higher utility value to x than to y.
[14] The convex hull defined by a set of vectors $\{p_i\}_{i=1,2\ldots}$ is the smallest convex cone of vectors which contains the set $\{p_i\}_{i=1,2\ldots}$. The Pareto condition is vacuously satisfied when the convex hull of $\{p_i]$ is empty.

Proof. See Chichilnisky [10].

Definition 7. A social choice rule $\phi : \prod_i P \to P$ is called dictatorial when there exists an individual $d \in I$ such that $\phi(\{p_i\}) = p_d \; \forall \{p_i\}$ in $\prod_i P$, i.e. when the map ϕ is the projection of the sequence $\{p_i\}$ onto its dth coordinate, ϕ is *non-dictatorial* if there is no d with $\phi(\{p_i\}) = p_d \; \forall \{p_i\} \in \prod_i P$.

Definition 8. A social choice $\phi : \prod_i P \to P$ *respects unanimity* when $\phi(\{p_i\}) = p$ whenever $p_i = p$ for all i.

Definition 9. A social choice rule $\phi : \prod_i P \to P$ is *anonymous* when

$$\phi(\{p_i\}_{i=1,2\ldots}) = \phi(\{p_{k_i}\}_{i=1,2\ldots})$$

where $\{k_i\}$ is any permutation of the set of integers $N = \{1, 2, \ldots\}$. An anonymous rule assigns the same outcome to a set of preferences independently of the order of the voters that have those preferences.

In order to define the social choice map we shall work with a particular compactification of the integers which is used in what follows to select continuously accumulation points in bounded sequences, the Stone–Cêch compactification, denoted βN. The following definitions of filters and their properties can be found in Choquet [19], Kelley [27] and Gillman [24].

Definition 10. A *filter* on a set X is a family F of subsets of X such that
(1) $U \in F,\; V \in F \to U \cap V \in F$
(2) $\phi \notin F$
(3) $U \in F,\; V \supset U \to V \in F$

Definition 11. An *ultrafilter* is a filter F such that there is no other filter $G \neq F$ with $G \supset F$.

A filter on X can be thought of as a family of neighborhoods of points for a topology of X, and an ultrafilter as a maximal such family. By the axiom of choice (Zorn's lemma) any filter is contained in a maximal filter, i.e. in an ultrafilter. An ultrafilter F for a set X is said to be *based on a point* $x \in X$ and is denoted $A(x)$ if it is the equivalent of the set of all neighborhoods of x in X, formally: $A(x)$ is based on the point x if the intersection of all subsets of X belonging to $A(x)$, is the point x.

Definition 12. An ultrafilter F is called *free* if there is no point x in X such tht F is based on x.

Example 13. If N is the set of integers, and B the filter of all subsets in N that are complements of finite sets, then B is called the Frechet filter and is contained in a maximal filter, called the *Frechet ultrafilter*[15]. This ultrafilter is free, i.e. it is not based on any integer in N.

Definition 14. The *Stone–Cêch compactification* of the set of integers N denoted βN is defined pointwise as the set of all ultrafilters of N. N is contained in βN by the identification of each point n in N with the ultrafilter based on n, i.e. the set of all subsets of N that contain the point n, see [27].

[15] B is also called the Fréchet filter, Choquet and Marsden [19].

Lemma 15. *The complement of N in βN (i.e. $\beta N - N$) is the set of all free ultrafilters of N. By the definition of the topology of βN, see Kelley [27], N is dense in βN. Therefore, any free ultrafilter of N can be approximated by ultrafilters based on integers in N. Furthermore, any continuous function $f: N \to Z$, where Z is a compact topological space, admits unique extension to a continuous function $\tilde{f}: \beta N \to Z$ which makes the diagram below commutative:*

$$\begin{array}{ccc} \beta N & & \\ i \uparrow & \searrow \tilde{f} & \\ N & \xrightarrow{f} & Z \end{array}$$

In this diagram, $i: N \to \beta N$ is the inclusion of N as a dense subset of βN.

Proof. This is the Stone–Cêch theorem: a proof is in Kelley [27, p. 153]. ■

The following provides another representation for the Sonte Cêch compactification. This definition is useful for our proofs because it explains the properties of "evaluation maps" which will be used in defining social choice rules. The following is a definition of the Stone Cêch compactification used in Kelley [27, p. 152], applied to the special case of the integers, N. Let $C(N)$ denote the space of all functions from the set of integers N into the unit interval I, i.e. a space of sequences of real numbers between 0 and 1. The spaces $I^{C(N)}$ is the product of I taken $C(N)$ times; it consists, by definition, of all functions $\alpha: C(N) \to I$, and it is compact with the product topology[16] by Tychonov's Theorem [27, p. 143].

For any $n \in N$ define the *evaluation map based on n*, $e_n \in I^{C(N)}$, by: $e_n(f) = f(n)$ $\forall f \in C(N)$. Now define the *evaluation map* $e: N \to I^{C(N)}$, as $e(n) = e_n$. Evaluation e is a continuous map of N into the product space $I^{C(N)}$ and defines a homeomorphism of N into a subspace of $I^{C(N)}$.

Definition 16. The closure of the image of N under the evaluation map e, denoted $cl(e[N])$, is a closed subset of the compact space $I^{C(N)}$ and, therefore, a compact space itself: this set $cl(e(N))$ together with the map e define a compactification of the integers N, βN. Since N can be identified with its image under the evaluation map, $e[N]$, it follows by construction that N is dense in βN.

The density of N in βN is used to extend functions from N into βN; this is Stone's theorem, stated above.

A profile of preferences in $\prod_i P$ defines a function from N into P. It follows from Stone's theorem and the fact that P is compact that:

Corollary 17. *A profile of preferences $\prod_i P$ defines a sequence of preferences in P, and therefore it admits a unique continuous extension to a function from ultrafilters to P, i.e. a function on βN to P, called a generalized profile of preferences.*

This result is used in the construction of the social choice rule in the next section.

[16] The product topology is the topology in which if $\{\alpha^j\} \subset P^{C(N)}$, $\alpha^j \to \alpha \in P^{C(N)} \Leftrightarrow \forall f \in C(N)$, $\alpha^j(f) \to \alpha(f)$.

Corollary 18. *Consider the space of generalized sequences of preferences $P^{\beta N}$, consisting of all the functions from βN into P, i.e. the product space of P with itself taken βN times. This space is compact.*

Proof. This is Tychonoff's theorem, see e.g. [27]. ■

Definition 19. Endow the space $P^{\beta N}$ with the product topology, namely with the topology of coordinate convergence-: a net $\{f_n, n \in D\}$ converges to $g \Leftrightarrow \{f_n(x), n \in D\}$ converges to $g(x)$ for each x in X[17]. $P^{\beta N}$ contains the infinite sequences of preferences in P, namely P^N, since every integer N is identified uniquely with an ultrafilter in βN.

Definition 20. Let ρ denote the product topology which the space of infinite sequences $P^N \subset P^{\beta N}$ inherits from its inclusion in the space $P^{\beta N}$ with its product topology.

From now on we shall work with the topology ρ on P^N. Note that the topology β is not the standard product topology on P^N; in fact it is finer, since it requires coordinatewise convergence not only on integers but also on all other elements of βN[18].

Lemma 21. *For any $F \in \beta N$, the evaluation map e_F defined as the projection on the coordinate $F \in \beta N$, namely, $e_F(f) = f(F)$ for all $f \in P^{\beta N}$ is a continuous map on $P^{\beta N}$ with its product topology. In particular, e_F defines a continuous map on P^N with the product topology ρ it inherits from its inclusion in $P^{\beta N}$.*

Proof. This is immediate; see e.g. Kelley [27, p. 218, lines 1–7]. ■

3. Construction of social choice rules for infinite populations

In this section we construct a social choice rule for infinite populations. This rule is continuous, Pareto and non-dictatorial: the social preference is the limit of the sequence of individual preferences, or some suitable generalization of the limit when the limit does not exist. Our strategy is as follows. First we establish our results for the simple case of profiles contained in a subset of $\prod_i P$ consisting of preference profiles that have limits with respect to the C^1 topology on vector fields defined above. Denote this by P_l:

$$P_l = \{p_k\} \in \prod_i P_i : \exists p \in P : \| p_k - p \| \to 0 \text{ as } k \to \infty\}.$$

The results are established very easily in this case. Theorem 1 then proves the results for the general case. An outline of the proof of this Theorem is given in this Section, with a complete proof being contained in the appendix.

[17] See e.g. Kelley [27, (3.4) and p. 217].

[18] For example, the sequence of preference profiles $\{\alpha_n\}_{n=1,2\ldots}$ defined by $\alpha_1 = (q, p_0, p, p, \ldots) \in P^N$, $\alpha_2 = (q, q, p, p, p, \ldots), \ldots,$ $\alpha_n = (\overbrace{q, q, \ldots, q}^{n-times}, p, p, p, \ldots)$ does not converge to the profile $\alpha = (q, q, \ldots, q, \ldots)$ in the product topology ρ on P^N, even though it does converge to α in the standard product topology of P^N.

We now turn to the case of preference profiles that have limits, i.e. those in P_l. P_l inherits the topology of $\prod_i P$, and is a closed subspace of $\prod_i P$. It is useful to note the following points:

- It follows from the topology on P that if $\{p^n\} \to p$ then $p^n(x) \to p(x)$ for any choice $x \in X$.
- Given the topology of $\prod_i P$, for all choices $x \in X$, $\lim_{m\to\infty} \{p_k\}^m_{k=1,2,\ldots} \to \{p_k\}_{k=1,2\ldots}$ implies that $\forall k = 1, 2, \ldots, \lim_{m\to\infty} \{p_k(x)\}^m \to p_k(x)$ in R^n.
- Even though P_l is a restricted domain for preference profiles, the normals of indifference surfaces of all possible preferences in P_l describe a complete sphere $S^{n-1} \subset R^n$ at each choice $x \in X$.

Lemma 22. *Let $\phi: P_l \to P$ be the social choice rule defined by*

$$\phi(p_1, p_2, p_3, \ldots) = \lim_k \{p_k\}.$$

Then ϕ is continuous, Pareto, non-dictatorial and respects unanimity and $\forall p \in P_l$, $\phi(p)$ defines a complete C^1 locally integrable social preference on the choice space X.

Proof. The map ϕ is well defined by the construction of the space P_l. Continuity of ϕ is immediate given the topology of P_l.

To see that ϕ is Pareto, note that if at any choice x in X the set of vectors $\{p_k(x)\}$ defines a spherical convex hull in S^{n-1}, i.e. if there exists some direction in which all preferences in $\{p_k\}$ increase, then since $\lim_k \{p_k(x)\}$ is contained in the closure of $\{p_k(x)\}$, it is certainly contained in the spherical convex hull of $\{p_k(x)\}$. Therefore by Lemma 1 in Section 2, ϕ is Pareto.

The map ϕ is not dictatorial because if only the dth-coordinate of a sequence in P_l is changed, for any $d = 1, 2, \ldots$, the limit of the sequence does not change. ∎

In order to extend the definition of ϕ overall of $\prod_i P$, we use the concept of limit derived from a free ultrafilter on the integers N.

Theorem 23. *There exists a continuous aggregation rule for infinite populations*

$$\psi: \prod_i P \to P$$

which is Pareto and non-dictatorial. ψ is constructed as an extension of the aggregation rule of Lemma 3 *above, and is a limit of dictatorial rules*[19].

Proof. The proof is in the Appendix. Its strategy is as follows. Using the limiting rule defined by a free ultrafilter F of the integers N, we assign to each profile (i.e. to each sequence of preferences in $\prod_i P$) another preference $p \in P$, as follows. The Frechet ultrafilter F assigns to the sequence of preferences $\{p_k\}$ an accumulation point of the set $\{p_k\}$ in P; since P is compact, $\{p_k\}$ has a non-empty set of accumulation points. Therefore the map $\psi\{p_k\})$ is well defined, and it corresponds to the limit of some subsequence of $\{p_k\}_{k=1,2\ldots}$.

[19] The map $\psi \in P^{\beta N}$, i.e., Ψ is in the space of continuous functions from the Stone–Cêch compactification of the integers βN (defined in Sect. 2) into the space of preferences P, and ψ is a limit of dictatorial rules when $P^{\beta N}$ is endowed with the product topology ρ.

By Arzela's theorem [19], this limit defines a continuous vector field. Finally, we show in the Appendix that ψ satisfies the continuity, Pareto, and non-dictatorship conditions, that ψ is a limit of dictatorial rules, and that, if the sequence $\{p_k\}$ has a limit, then $\psi\{p_k\} = \lim_K\{p_k\}$, and therefore ψ coincides with the rule ϕ when restricted to the subset P_l of $\prod_i P$, consisting of sequences that have a limit.

4. Asymptotic dictators with compactified populations

The construction of Theorem 1 shows that the social choice rule is a limit of dictatorial rules; indeed, the proof of Theorem 1 shows that the social choice rule has an "asymptotic dictator", which is an element of the compactified set of individuals. Here we show that the rule we constructed has the property that, when the set of integers is given a finite measure, the asymptotic dictator is "in coalition with" sets of voters of arbitrarily small measure. In this sense, sets of voters of arbitrarily small measure are decisive.

Definition 24. The compactified space of preferences profiles is $P^{\beta N}$ when the space of individuals is the Stone–Cêch compactification of the integers, i.e. βN, as defined in Sect. 2.

Theorem 25. *Assume that the infinite set of voters N is compactified. The rule ψ constructed in Theorem 1 is in this case a dictatorial rule with dictator F, where F is the element of βN used to define ψ, and F is a limit of dictators in the (non-compactified) infinite set of voters N, i.e. it is an asymptotic dictator.*

Proof. In the Appendix.

Definition 26. We say that a subset of individuals $M \subset N$ controls the social choice rule ψ when $\psi(\{p_i\}) = \psi(\{q_i\})$ for every profile $\{q_i\}$ which only differs from $\{p_i\}$ in the preferences of individuals outside the set M, i.e., whenever

$$\forall i \in M, \quad p_i = q_i \Rightarrow \psi(\{p_i\}) = \psi(\{q_i\}).$$

Definition 27. We say that *a dictator D is in coalition with a controlling set of individuals M* if Ψ is dictatorial with dictator D, and Ψ is controlled by the subset M.

In the following we shall assume that the set of voters is given a finite measure μ.

Definition 28. A finite measure on the set of voters is a σ-additive function μ defined on the subsets of the set of integers N, such that $\mu(N) < \infty$.

Corollary 29. *If the space of voters is compactified, then the asymptotic dictator of the continuous Pareto rule ψ is in coalition with controlling sets of voters of arbitrarily small measure.*

Proof. By definition

$$\psi\{p_k\} = ev_F\{p_k\},$$

where F is a free ultrafilter in N, such as the Frechet ultrafilter defined in Sect. 2. Since F is free, it is a limit of a sequence of ultrafilters based on integers, i.e., is based on a sequence of integers $\{j_n\}_{n=1,2\ldots} \to F$ in βN, where the sequence $\{j_n\}_{n=1,2\ldots}$ is unbounded. Therefore by definition, the social choice rule Ψ is controlled by subsets of individuals M^T which are complements of arbitrarily large finite sets of integers $T = \{1, 2, \ldots, T\}$, where $M^T = N - T$. Since the measure μ is finite, $\forall \varepsilon > 0\ \exists T$: $\mu(M^T) < \varepsilon$, thus completing the proof. ■

5. Continuity, independence of irrelevant alternatives, and previous results

The results of the last section provided, for infinitely many voters, a continuous social choice rule that satisfies the Pareto axiom and is non-dictatorial. The construction assigns to each profile of preferences a social preference given by a preferred direction at each point and determining a locally integrable vector field. Therefore, a well defined socially preferred direction is obtained at each choice, and in this sense our aggregate social preference is complete.

Previous work by Fishburn [23] and Kirman and Sonderman [26] studied existence of social choice rules that are Pareto, non-dictatorial and satisfy the axiom of independence of irrelevant alternatives for infinite populations. However, even though both [23] and [26] prove existence results, the social preference whose existence they prove need only be a *weak order*. The problem is that *weak orders* may be incomplete, or even completely trivial. For example, the empty set defines a weak order, so that their results may be empty. Neither [23] or [26] rule out those case so that those results may leave the social choice paradox unresolved in some cases. In our case, the social preference is always complete and not trivial. There is, then, a difference in the "strengths" of the results presented here and those earlier results.

In addition, the approaches of [23] and [26], when applied to continuous choice spaces, may lead to a social preference which is discontinuous, even though all individual preferences are continuous. The demonstration of this point helps to clarify the difference between the approaches. These authors consider a social preference defined by using an ultrafilter as a family of decisive sets. In other words: a choice x is preferred to another y when all individuals in the sets of the ultrafilter prefer x to y.

Fig. 2 depicts preferences at a point x in the choice space, assumed to be a subset of the plane. Each preference is identified by the normal to the indifference surface at x: we show a sequence of preferences $p_1, p_2, p_3, \ldots, p_n, \ldots$, which converge to the preference p_0. As preferences are smooth, the indifference surfaces can be approximated locally by straight lines. So if y is near x, x is preferred to y for every p_i $i = 1, 2, 3, \ldots$, but for p_0, x is indifferent to y.

This construction enables us to bring out the difference between the two approaches. According to the approach of this paper, the social preference at x is clearly p_0. Hence x is socially indifferent to y. However, as x is preferred to y by all individual preferences, there must be a unanimous preference for

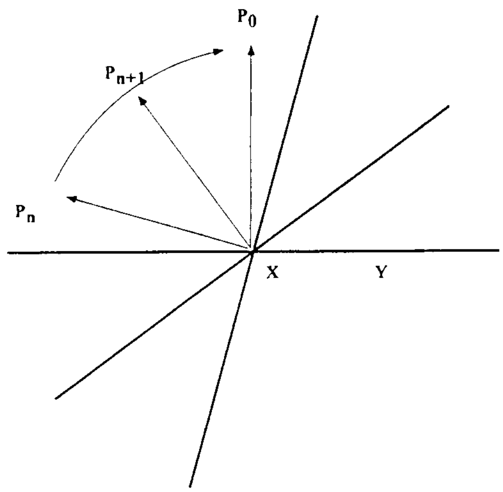

Fig. 2. A sequence P_n of linear preferences converging to P_0. X is preferred to Y by all preferences in the sequence, but for the limiting preference they are indifferent

x over y by some set in any free ultrafilter over the agents, so that according to the approaches of [23] and [26], x is socially preferred to y.

We shall now demonstrate the discontinuity of the social preference derived by using an ultrafilter as a family of decisive sets as done by [23] and [26]. Consider a point z near to and above y. Clearly there exists an $n(z)$, depending on z, such that for all preferences n, $n \geq n(z)$, z is preferred to x. It follows that any free ultrafilter will contain a set of agents who all prefer z to x, so that according to the approach of [23] and [26], z is socially strictly preferred to x. Letting p denote the social preference according to these approaches, we have:

$$z\,p\,x \quad \text{and} \quad x\,p\,y.$$

Now the argument that gives $z\,p\,x$ works for any z above y. Hence we can construct a sequence z_i, converging on y from above, everyone satisfying $z_i\,p\,x$. Thus the z_i satisfy:

$$\lim_{i \to \infty} z_i = y, \quad z_i\,p\,x \text{ all } i, \text{ and } x\,p\,y.$$

It follows that the social preference constructed by [23] and [26] is discontinuous. In particular, it cannot be represented by a gradient vector at x.

Another significant difference of our results with those of [23] and [26] is that our *social choice rule is continuous* on the preferences of the voters, so a small change in individual preferences leads to only a small change in the social preferences. Instead, [23] and [26] require independence of irrelevant alternatives, and do not establish continuity. There is a significant difference between the condition of continuity and the axiom of independence. Independence requires that the outcome *on any pair of choices* be the same if the individual preferences are changed outside of those choices. In order to define the continuity condition, individual preferences must be continuous, and a continuous preference cannot be arbitrarily changed outside of two choices without affecting the preference on (infinitesimally) close choices also.

Therefore, continuity rules out some of the conditions under which independence is tested, since arbitrary changes outside of two choices are not always possible. However, our rule is defined by finding a most preferred direction at each choice, as a function of the most preferred directions and their rates of change, for the individual preferences, at those choices. It follows that outside any small ball of radius say for any $\varepsilon > 0$, around each choice, our rule is independent of irrelevant alternatives. Independence in our framework can, therefore, be defined as follows.

- **Global Independence Axiom:** The outcome of the social rule

$$\psi : \prod_i P \to P$$

at any choice x is the same when the individual preferences in $\prod_i P$ are varied outside of an arbitrarily small neighborhood of x.

And our Theorem 1 in Sect. 3 can now strengthened now as follows:

Corollary 30. *There exists a continuous aggregation rule for infinite populations*

$$\psi : \prod_i \bar{P} \to P$$

which is Pareto, non-dictatorial and satisfies the Global Independence Axiom. ψ is constructed as in Theorem 1.

[23] and [26] showed that even though a social choice rule exists, there are still what they term "invisible dictators", so that the resolution of the paradox is more apparent than real in the infinite agent case. Our approach to the problem of social choice with infinitely many agents, has a more straightforward intuitive basis than that of [23] and [26]. Both of these earlier contributions use a non-constructive proof of the existence of a Pareto, non-dictatorial social choice rule: [23] relies on Zorn's Lemma, and [26] use equivalent results. In each case, what is shown is just the existence of a rule: the rule is not characterized.

Instead, here we tackle both the questions of existence and of characterization. Our proof of existence is quite different form, and intuitively more straightforward than, the earlier ones, and also provides a characterization of the kind of rule that it produces. When the rule discussed by [23] and [26] is reinterpreted within the present framework of preferences over Euclidean choice spaces, it turns out to have very different properties from the one discussed here, as pointed out above.

6. Conclusions

We constructed, for infinite populations, social choice rules which are continuous, Pareto and non-dictatorial. Our rules are furthermore globally independent of irrelevant alternatives. These rules were constructed by taking the limit of the sequence of individual preferences, when this exists, and otherwise using an appropriate generalization. It is clear from their construction that these rules may not be anonymous, and that their being non-dictatorial relies in an essential way on there being an infinite population. In fact we showed that these rules are limits of dictatorial rules. For each finite population, the last person is dictator: the problem is solved for infinite populations because in the infinite case, there is no last person.

7. Appendix

This appendix contains the proofs of Theorems 1 and 2 above.

Proof of Theorem 1. Let $\{p_i\}_{i=1,2...}$ be a profile of preferences, i.e. a sequence of preferences in $\prod_i P$; this profile is a map from integers to preferences, and therefore can be written also as a function $f^{\{p_i\}}: N \to P$. By Stone's theorem, stated in Sect. 2, since P is compact, there exists a unique extension of $f^{\{p_i\}}$ to a continuous map $\tilde{f}^{\{p_i\}}$ defined on βN,

$$\tilde{f}^{\{p_i\}}: \beta N \to P.$$

By construction the map $\tilde{f}^{\{p_i\}}$ assigns continuously to each ultrafilter in βN a preference in P, in such a way that if $n \in N$ is identified with the ultrafilter based on n, then $\tilde{f}^{\{p_i\}}(n) = f^{\{p_i\}}(n)$. Continuity refers here to the topology of βN, as defined in Sect. 2.

We now define a social choice rule. Recall the definition of the evaluation map based on the ultrafilter F given in Sect. 2: it maps any function $g: \beta N \to P$ into its value at F, namely $e_F(g) = g(F) \in P$. In particular if $n \in N$ is identified with the ultrafilter based on n, then the evaluation map e_n simply maps the function $g \to e_n(g) = g(n) \in P$.

We use an evaluation map based on a free ultrafilter F of βN, such as for example the Frechet ultrafilter defined in Sect. 2, in order to define the social choice rule. Define:

$$\psi: \prod_{i=1}^{\infty} P \to P \quad \text{as } \Psi(\{p_i\}) = e_F(\tilde{f}^{\{p_i\}}), \tag{1}$$

where the map $\tilde{f}^{\{p_i\}}: \beta N \to P$ in (1) is the unique Stone extension of the map $f^{\{p_i\}}: N \to P$ which defines the preference profile $\{p_i\} \in \prod P$, as defined above.

We now check that the map ψ has all the desired properties. First, continuity. The evaluation map e_F is continuous map on the space of maps $g: \beta N \to P$ into P[20]. Therefore e_F is also continuous on the restriction maps $g/N: N \to P$, which are the profiles of preferences, with the product topology[21].

[20] See Kelley [27, p. 116, Embedding Lemma 4.5]. The continuity is with the product topology.

[21] Kelley [27, p. 152, paragraph 2], considers a topological space X and calls $F(X)$ the family of all continuous functions on X to the closed unit interval Q. The evaluation map e carries a member $x \in X$ into the member $e(x)$ of the cube $Q^{F(x)}$, whose f-coordinate is $f(x)$ for each $f \in F(X)$. He states that "evaluation is a continuous map of X into the cube $Q^{F(X)}$". The topology he uses on $Q^{F(X)}$ is the product topology. In our case $Q = P$, the compact metric space of preferences P, and $X = \beta N$, the Stone Cech compactification of the integers, as defined in Section 2.

Secondly, we prove that the map ψ is Pareto. Since N is dense in βN, the ultrafilter F is a limit in the space βN of ultrafilter based on integers, i.e. $F = \lim_{j\to\infty}\{n^j\}$. By continuity of the evaluation map, $e_F = \lim e_{n^j}$, so that $e_F(f^{\{p_i\}}) = \lim_j e_{n^j}(f^{\{p_i\}}) = \lim_{j\to\infty}\{p_{n^j}\}$, so that $\psi(\{p_i\})$ is an accumulation point of the sequence of vectors $\{p_i\}$. Since $\lim_{j\to\infty}\{p_{n^j}\}$ is an accumulation point of $\{p_i\}$, then, in particular, if the convex hull $\{p_i\}$ is not empty, $\psi(\{p_i\})$ is contained in this convex hull. Therefore by Lemma 1 in Sect. 2, ψ is Pareto.

Next we verify that ψ is not dictatorial; this follows directly from the fact that $e_F \neq e_n$ for any $n \in N$, since the ultrafilter F is a free ultrafilter. We have therefore proved that ψ is continuous, Pareto and non-dictatorial.

Finally, we show that the map ψ is a limit of dictatorial maps. Recall that N is dense in βN, so that the ultrafilter F can be approximated by a net of integers in N: i.e. in the topology of βN, $F = \lim_{j\to\infty}\{n^j\}$. The evaluation map e which maps any function $g:\beta N \to S^{n-1}$ into its value at F, namely $e_F(g) = g(F) \in S^{n-1}$ is continuous, see Kelley [27][22], so that:

$$e_F = e_{\lim\{nj\}} = \lim e_{n^j}.$$

Since e_{n^j} is, by definition, the map that assigns to a sequence of preferences its n^jth coordinate, it follows that e_{n^j} is dictatorial. Therefore, ψ is the limit of dictatorial rules, as we wanted to prove. Finally, since the integers $\{n^j\}$ that approximate F in βN cannot converge to a finite limit (for, otherwise, F would not be free) it follows that when the sequence $\{p_k\}$ has a limit, then $e_{n^j}(\{p_k\}_{i=1,2\ldots})$ converges to $\lim_{k\to\infty}\{p_k\}$. This implies that ψ restricted to P_l coincides with the social choice rule ϕ defined in Theorem 1. ■

Proof of Theorem 2. This is an immediate corollary from Theorem 1, defining the ultrafilter F as an "asymptotic dictator". ■

References

1. Arrow KJ (1963) Social Choice and Individual Values, 2nd ed. Wiley, New York
2. Baryshnikov (1993) Unifying impossibility theorems: a topological approach. Adv App Math 14: 404–415
3. Black D (1948) The rationale of group decision-making. J Polit Econ 56: 23–24
4. Candeal JC, Chichilnisky G, Indurain E (1997) Topological aggregation of preferences: the case of a continuum of agents. Soc Choice Welfare 14: 333–343
5. Candeal JC, Indurain E, Uriarte JR (1992) Some issues related to the topological aggregation of preferences. Soc Choice Welfare 9: 213–227
6. Chichilnisky G (1976) Manifolds of Preferences and Equilibria. Report No. 27, Project on Efficiency Decision Making in Economic Systems. Harvard University
7. Chichilnisky G (1980) Social choice and the topology of spaces of preferences. Adv Math 37(2): 165–176

[22] See Kelley, [27, p. 116, Embedding Lemma 4.5 and p. 152, paragraph 2]: If X is a topological space and $F(X)$ is the family of all continuous functions from X into the closed unit interval Q, the evaluation map e which asigns to each member $x \in X$ the member $e(x) \in Q^{F(X)}$ whose f-coordinate is $f(x)$ for each $f \in F(X)$ is a continuous map from X into $Q^{F(X)}$. In our case, X is the Stone Cech compactification of the integers, βN, defined in Sect. 2, and Q is the space of preferences P.

8. Chichilnisky G (1982) Social aggregation rules and continuity. Q J Econ 1982: 337–352
9. Chichilnisky G (1982) Structural instability of decisive majority rules. J Math Econ 9: 207–221
10. Chichilnisky G (1982) The topological equivalence of the Pareto condition and the existence of a dictator. J Math Econ 9: 223–233
11. Chichilnisky G (1994) Social diversity, arbitrage and gains from trade: a unified perspective on resource allocation. Am Econ Rev 84(2): 427–434
12. Chichilnisky G (1993) On strategic control. Q J Econ 1993: 286–290
13. Chichilnisky G (1995) Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium. Econ Theory 5(1): 79–108
14. Chichilnisky G (1993) Markets, arbitrage and social choice. [Department of Economics, Columbia University, 1991] CORE Discussion Paper No. 9342
15. Chichilnisky G (1995) A Unified perspective on resource allocation: Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, the core and social choice. In: Arrow K, Sen A, Suzamura K (eds) The Theory of Social choice. International Economic Association, McMillan
16. Chichilnisky G (1981) Markets, Games and their Topology
17. Chichilnisky G (1994) What is sustainable development? Stanford Institute for Theoretical Economics Technical Report No. 65
18. Chichilnisky G, Heal GM (1983) Community preferences and social choice. J Math Econ 12: 33–61
19. Choquet G, Marsden J (1969) Lectures on Analysis. Benjamin, Amsterdam, New York
20. Debreu G (1972) Smooth preferences. Econometrica 40: 603–615
21. Dunford N, Schwartz J (1966) Linear Operators (Part I). Interscience, New York
22. Effimov BA, Koshevoy GA (1993) The topological approach to social choice with infinite populations. Math Soc Sciences 27(2): 145–157
23. Fishburn PC (1970) Arrow's impossibility theorem, concise proof and infinite voters. J Econ Theory 2: 103–106
24. Gillman L, Jerison M (1960) Rings of Continuous Functions. Van Nostrand, Princeton, NJ
25. Hirsch MW (1977) Differential Topology. Springer, Berlin Heidelberg New York
26. Kirman A, Sondermann D (1972) Arrow's theorem, many agents and invisible dictators. J Econ Theory 5: 267–277
27. Kelley J (1955) General Topology. (University Series in Higher Mathematics) Van Nostrand, Princeton, NJ
28. Lauwers L (1997) Infinite Chichilnisky rules. Soc Choice Welfare 14: 345–356
29. Lauwers L (1997) A Note on weak ∞-rules. Soc Choice Welfare 14: 357–358
30. Lauwers L, van Liedekerke L (1993) Monotonic Chichilnisky rules with infinite populations. Working Paper, Monitoraat E.T.E.W. Katholiek Universiteit Leuven, Leuven, Belgium
31. Neveu J (1965) Mathematical Foundations of the Calculus of Probability. Holden Day, San Franscisco
32. Spanier E (1966) Algebraic Topology. (McGraw-Hill Series in Higher Mathematics) New York