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On the existence and the structure of the pseudo-equilibrium manifold

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Abstract

A brief proof of the existence of an equilibrium in incomplete markets is given for regular economies (Theorem 1); a robust example of such economies is provided. Theorem 2 establishes that the manifold Ω of global pseudo-equilibria is a fiber bundle with the topological structure of a one- or two-fold covering space of $G^{k,n}$, the space of k planes in R^n, where k is the number of assets in the economy and n the number of states of uncertainty. In particular, the pseudo-equilibrium manifold Ω is not contractible with incomplete markets. This contrasts with the case of complete markets, where the equilibrium manifold is always contractible.

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1. Introduction

Two fundamental problems in economics are the existence of a market equilibrium and the structure of the manifold of equilibria. Both have been resolved satisfactorily in standard Arrow–Debreu markets: simple proofs of existence have been provided, and the equilibrium manifold is known to be contractible (Balasko, 1975).

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Matters are quite different in markets that have an incomplete set of assets. All known proofs of the existence of an equilibrium are rather complex, and nothing is known of the structure of the pseudo-equilibrium manifold. [1] To extend the classical case, this paper provides a brief proof of existence of an equilibrium in markets with an incomplete set of assets (see Daffie and Shafer, 1985, and Radner, 1972), and determines the global topological structure of the pseudo-equilibrium manifold in such markets.

Theorem 1 provides a brief proof of existence of a pseudo-equilibrium. Theorem 2 studies the pseudo-equilibrium manifold and establishes a substantial departure from the structure of equilibria in economies with complete markets. When markets are complete, the equilibrium manifold is always topologically trivial. By contrast, we demonstrate in Theorem 2 that with incomplete markets the pseudo-equilibrium manifold Ω is in general not contractible, and provide examples.

Our proof of existence of a pseudo-equilibrium is similar to standard proofs for proving the existence of an equilibrium with complete markets. To achieve this, the proof of Theorem 1 relies on the existence of a continuous selection from asset returns to spot prices; [2] Proposition 2 establishes that such a selection exists in *regular* economies. [3] The appendix provides a robust example of regular economies with incomplete asset markets which satisfy all the conditions in Theorem 1. Our proof of existence therefore applies to a robust set of economies.

Theorem 2 gives a topological characterization of the manifold of pseudo-equilibria: this is a more demanding task than the proof of existence, requiring algebraic topology. Theorem 2 proves that the pseudo-equilibrium manifold Ω is a fiber bundle over the manifold of k-dimensional subspaces of R^n, denoted $G^{k,n}$, where n is the number of states and k is the number of assets, [4] and that Ω is topologically equivalent to a one- or two-fold covering space of $G^{k,n}$, and is equal

[1] The pseudo-equilibrium manifold is equivalent to the equilibrium manifold in Arrow–Debreu markets. It describes all the pseudo-equilibrium spot prices and asset returns which obtain by varying the economy's initial endowments and real assets; see Subsection 4.3.

[2] Our existence result does not add generality to the proofs of existence available. Rather, it aims to show that, in regular economies, proving the existence of an equilibrium with incomplete markets is similar to proving existence in complete markets. The only addition required is a fixed-point-like theorem on subspaces of R^n.

[3] Regular economies with incomplete markets are a natural extension of the concept of regular economies defined by Debreu for Arrow–Debreu (complete) markets. The condition required in Lemma 1 is that a Jacobian of the excess demand function is non-zero at the set of prices and asset spaces which clear the markets. When markets are complete, this is simply Debreu's condition of regular economies.

[4] The space of all k-dimensional subspaces of n-euclidean space is also called the (k, n) Grassmanian manifold and is denoted $G^{k,n}$. After refereeing and editing this paper for the *JME*, Wayne Shafer extended our results to study the orientability of the pseudo-equilibrium manifold, and the structure of the manifold for a given, fixed asset structure, in a note 'Note on the pseudo-equilibrium manifold', December 1993.

to either $G^{k,n}$, or to the manifold $M^{k,n}$ of oriented k planes in R^n. These two are classical manifolds with a relatively well-known topological structure: neither is contractible in general, and we provide examples. Therefore the pseudo-equilibrium manifold Ω is generally not contractible. When the market is complete $(k=n)$ the standard result is recovered: Theorem 2 implies the contractibility of the equilibrium manifold in Arrow–Debreu markets because $G^{n,n}=M^{n,n}=R^n$.

2. Economies with incomplete asset markets

2.1. Definitions and notation

An economy has m *commodities* and two *periods* $(t=1, 2)$. Uncertainty is represented by n *states* in the second period. In period 1 there are m *spot markets*, and securities markets for *assets* that pay bundles of m commodities in period 2.[5] An *economy* E consists of J *traders*, indicated $g=1\dots J$, each with a *utility function* U_g over consumption in both periods, an *initial endowment* w_g in the interior of the positive orthant in R^r, denoted R^r_{++}, where $r=m\times(n+1)$, and an *asset structure* $a=(a^1,\dots,a^k)\in R^{k\times m\times n}$, which is a collection of k real assets, $k\le n$, each asset denoted $a^i\in R^{m\times n}$, $1\le i\le k$.[6]

There are $r=m\times(n+1)$ spot markets. For every spot price $p\in\Delta=\{p\in R^r: p\gg 0 \text{ and } \|p\|=1\}$, the asset structure enables us to calculate the value at that price p of the asset i in state s, $a^i(s)$, i.e. $\langle p_s, a^i(s)\rangle$. An *asset return matrix* consists of k vectors $\{v_i\}$ of dimension n each, vector v_i denoting the value at prices p which one unit of the asset gives in each state $s=1\dots n$ in period 2, $v_i=v_i(1)\dots v_i(n)$, where $\forall i=1\dots k$, $v_i=\langle p_s, a^i(s)\rangle$. The vectors $\{v_i\}_{i=1\dots k}$ define a subspace of R^n, denoted L. We denote by $[v_1,\dots,v_k]$ or simply $[v_i]_{i=1\dots k}$ the linear span of the vectors $\{v_i\}_{i=1,\dots,k}$. Given an asset structure a, for each vector of spot prices $p\in\Delta$ there is an associated space of asset returns $L(p, a)\subset R^n$ of dimension at most k. Let $R^{k,n}$ denote the space of all k linearly independent vectors in R^n, also called 'k frames'.

Definition 1. We say that a map $f:\Delta\times R^{n\times k}\to R^{m\times(n+1)}$ is *homogeneous* on $R^{n,k}$ if $f(p,[v_1,\dots,v_k])=f(p,[w_1,\dots,w_k])$ whenever the sets of vectors $\{v_i\}$ and $\{w_i\}$ have the same span L, i.e. when $[v_1,\dots,v_k]=[w_1,\dots,w_k]=L$, in which case we denote the map $f(p, L)$.[7]

[5] Called real or physical assets.

[6] A unit of *asset* a^i entitles the holder to the vector $a^i(s)\in R^m$ of commodities in period 2 if the state $s\in\{1\dots n\}$ occurs.

[7] Continuity of a homogeneous function f on $R^{n\times k}$ is defined with respect to the standard proximity criterion: the distance between two spaces L of R^n is the infimum of the distances between all orthonormal bases for the two spaces.

Given initial endowments w_g, an asset structure a, and utilities U_g, the *excess demand* is defined as a homogeneous function of spot prices p and of the corresponding asset returns L. In an *equilibrium*, all prices and returns must be compatible with each other, and all markets must clear: therefore an equilibrium is a vector of spot prices $p^* \in \Delta$ and a space of asset returns $L^* \subset R^n$, at which the excess demand in all the spot markets is zero, and the equilibrium asset returns L^* are those corresponding to the equilibrium spot prices p^*.

2.2. A standard economy with incomplete assets

It is helpful to formalize the economy E in a standard abstract fashion (see, for example, Hirsch et al., 1990), considering the excess demand function and the asset structure as the primitive concepts. The economy is therefore characterized by:

- An *excess demand function*: this is a continuous homogeneous function $Z: \Delta \times R^{n,k} \to R^r$, assigning to each spot price $p \in \Delta$ and each k-dimensional space $L = [v_1, \dots, v_k] \subset R^n$ of asset returns the excess demand vector of the economy in all spot markets, and satisfying:
  (i) $p.Z(p, L) = 0$ for all $p \in \Delta$ (Walras' Law);
  (ii) there is $b \in R^r$ such that $Z(\Delta \times R^{n,k}) \geq b$, i.e. excess demand is bounded below;
  (iii) If $(p_m, L_m) \in \Delta \times R^{n,k}$, $(p_m, L_m) \to (p, L)$ and $p \notin \Delta$, then $\| Z(p_m, L_m) \| \to \infty$, i.e. when a spot price goes to zero, the norm of the excess demand vector increases beyond any bound.
- *k asset return functions* $f_j : \Delta \to R^n$, $1 \leq j \leq k$, describing for each set of spot prices, the returns of the k assets in the n states, which is any continuous function $f: \Delta \to R^n$ with $f = (f_1 \dots f_k)$, where $f_i : \Delta \to R^n$.

The economy E is therefore defined as:

- $E = \{Z: \Delta \times R^{n,k} \to R^r$ satisfying (i), (ii) and (iii), and a continuous function $F: \Delta \to R^n$,
  $f = (f_1 \dots f_k)$, where $f_i : \Delta \to R^n\}$.

3. Existence of a pseudo-equilibrium

We prove the existence of a pseudo-equilibrium in two steps. The first step is to show the existence of a vector of spot prices p which clears all markets for each given asset returns space L. This establishes a correspondence from asset returns to spot prices. Under the additional assumption that this correspondence has a continuous selection, the existence of a pseudo-equilibrium is established in a simple fashion in Theorem 1 below. The usefulness of this proof hinges of course on the existence of a continuous selection from spaces L to spot prices p. We prove in Proposition 2 below that such a continuous selection exists in every regular economy, and in Lemma 1 we prove that the property of being a regular economy is robust, i.e. it is satisfied on an open set of excess demand functions which define the economy. Therefore our proof of existence is valid for a robust set of economies.

3.1. What is a pseudo-equilibrium?

Consider the economy E defined in Subsection 2.2. When $k < n$ there are fewer assets than states, and E is called an economy with *incomplete* asset markets. A pseudo-equilibrium is a vector of spot prices p^* at which all markets clear, and such that the net trades at the prices p^* are feasible at the asset returns arising from these spot prices. When markets are complete this definition coincides with that of a competitive equilibrium. Formally:

Definition 2. Given the excess demand function $Z: \Delta \times R^{n,k} \to R^r$, and the asset return functions $f_j: \Delta \to R^n$, a *pseudo-equilibrium* is a spot price vector p^* in Δ, and k asset returns $v_1^*, \ldots, v_k^*$ spanning a subspace L^* of R^n, such that: (a) spot markets clear at the spot prices p^* and assets returns L^*, and (b) the returns on the assets which arise from the spot prices p^*, $v_1 = f_1(p^*), \ldots, v_k = f_k(p^*)$, are contained in L^*.

It is worth noting that part (b) of the definition of a pseudo-equilibrium does not require that the span of the vectors $v_1, \ldots, v_k$ should equal L^*, but rather that it should be contained in L. The intuition behind this definition (Duffie and Shafer, 1985) is that generically on endowments and return matrices, the vectors $v^1 = f_1(p^*), \ldots, v^k = f_k(p^*)$ are linearly independent, so that their span is actually equal to L^*.

3.2. A simple result on spot prices

Proposition 1 below establishes the existence of a vector of market-clearing spot prices p^* for each fixed asset returns space $L \subset R^n$, namely a vector of spot prices satisfying part (a) of the definition of a pseudo-equilibrium. Proposition 1 is not a proof of existence because to prove existence one must establish not only (a) but also (b): the full existence proof is given in Theorem 1 below. Problem (a) is identical to that of establishing the existence of a price equilibrium in complete markets, and it is proven with the same standard techniques:

Proposition 1. Consider an economy with incomplete asset markets E. Under assumptions (i), (ii), and (iii) on the excess demands, for each k-dimensional space of asset returns $L \subset R^n$, there exists a spot price $p^*(L)$ which clears spot markets, i.e. such that $Z(p^*(L), L) = 0$.

Proof. By assumptions (i), (ii) and (iii), for each given L, the excess demand function $Z(p, L): \Delta \to R^r$ is continuous, there exists a b in R^r such that $Z(\Delta, L) \geq b$, and furthermore if $(p_m, L) \to (p, L)$ with $p \notin \Delta$, then $\| Z(p_m, L) \| \to \infty$. Under these conditions, the existence of an equilibrium price $p^*(L)$ is ensured by standard arguments; see, for example, Theorem 8 in Debreu (1982). □

Proposition 1 therefore establishes the existence of a non-empty correspondence $\Psi: R^{n,k} \to \Delta$, which assigns to each $(v_1, \dots, v_k) \in R^{n,k}$, a price p in Δ at which markets clear and where agents face asset returns given by $L = [v_i]$; Ψ is clearly homogeneous, i.e. $\Psi([v_1, \dots, v_k]) = \Psi([w_1, \dots, w_k])$ if $[v_i] = [w_i]$.

3.3. Regular economies

In certain economies Proposition 1 can be sharpened to the existence of a *function*, rather than a correspondence, from asset return spaces L to equilibrium spot prices $p(L)$. We shall consider the following property, which is proven to be robust in Proposition 2 below:

(iv) There exists a continuous homogeneous map $\Gamma: R^{n,k} \to R^r$ assigning to each k-dimensional subspace $[v_1, \dots, v_k] = L$ of R^n, an equilibrium price $\Gamma(L)$ in Δ. Such a function is a continuous selection of the correspondence Ψ of Proposition 1.

Property (iv) is automatically satisfied when the economy E has a complete set of assets ($k = n$).[8]

When does an economy with incomplete markets satisfy condition (iv)? Proposition 2 below answers this question: it shows that condition (iv) is always satisfied when the excess demand function Z of the economy varies sufficiently with changes in spot prices. Such economies are called regular economies, and are defined formally as follows.

Definition 3. An economy is called *regular* when $S = \{(p, L) \subset \Delta \times R^{n,k}: Z(p, L) = 0\}$ is a manifold, and the Jacobian of the excess demand function Z with respect to spot prices p, $\partial Z(p, L)/\partial p$, contains an $(r-1) \times (r-1)$ submatrix with non-zero determinant $\forall (p, L) \in S$. In complete markets, i.e. when $k = n$, this is the standard definition of a regular economy.

Arrow–Debreu economies are regular generically on endowments; this result was established by G. Debreu and S. Smale in 1970. Since the economy E is defined in subsection 2.2 in terms of its excess demand function Z, it is appropriate to refer to the assumption of regularity in terms of excess demand.

[8] In this case the space of all subspaces of dimension n in R^n consists of one element, namely $\{R^n\}$.

Lemma 1. The property of being a regular economy is robust with respect to excess demand functions.

Proof. The value {0} is a regular value of the smooth map $Z: \Delta \times R^{k,n} \to R^r$ generically on excess demand functions; this follows from standard transversality theory [9] (see also Duffie and Shafer, 1985). Therefore S, which is the inverse under Z of {0} is a manifold generically on Z: this is the global version of the implicit function theorem (see Abraham and Robbins, 1967). An economy is regular when the Jacobian of its excess demand function with respect to the variable p is transversal to S, a property which is clearly open on excess demand functions. □

Proposition 2. A regular economy E satisfies property (iv).

Proof. See the appendix.

3.4. The existence of a pseudo-equilibrium

The next theorem provides the proof of existence of a *pseudo-equilibrium.*

Theorem 1. Consider an economy E with incomplete asset markets, i.e. $k < n$. *Under conditions (i), (ii), (iii), and (iv) there exists a pseudo-equilibrium for E.*

Proof. Condition (iv) ensures that there exists a continuous homogeneous function $\Gamma: R^{n,k} \to \Delta$ from asset returns L to spot prices $p \in \Delta$, $L \to p(L)$, such that $Z(\Gamma(L), L) = 0$. Consider now the k asset return functions (see Subsection 2.2) $f_1, \ldots, f_k$, $f_i: \Delta \to R^n$, and define the composition maps

$$g_i = f_i \circ \Gamma,\ g_i: R^{n,k} \to R^n. \tag{1}$$

We now use the following 'fixed subspace property': If $h_j: R^{n,k} \to R^n$, $1 \le j \le k$, are continuous homogeneous functions, then there is always a k-dimensional subspace $L = [v_1 \ldots v_k]$ of R^n such that $\forall j$, $h_j(L) \in L$. [10] Since the maps $g_i: R^{n,k} \to R^n$, $i = 1, \ldots, k$, defined in (1) are k continuous homogeneous maps, by the 'fixed subspace property' there is a k-dimensional subspace L^* of R^n such that $g_i(L^*) \in L^*$ for all i. The corresponding vector $(\Gamma(L^*), L^*)$ satisfies the definition of a pseudo-equilibrium, completing the proof of existence. [11] □

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[9] The space of smooth demand functions from $\Delta \times R^{k \times n} \to R^r$ is endowed with the standard compact open topology. In this context a *property* is called *robust* when it applies to an open set. For more details on both issues see Abraham and Robbins (1967).

[10] See Theorem 2 in Hirsch et al. (1990), or Theorem A in Husseini et al. (1990).

[11] This method for proving existence was suggested, but not executed, in Husseini et al. (1990). They state that ''they had no hope to prove continuity of spot prices as a function of asset returns'', an issue which is resolved in this paper, see Proposition 1 and the proof in the appendix.

4. The structure of the pseudo-equilibrium manifold

In this section we characterize the topological structure of the pseudo-equilibrium manifold Ω, and show that it is not contractible.

4.1. Definitions

Each real asset matrix a can be represented by a vector a in $R^{k\times m\times n}$; there are k assets, indicated by $i = 1 \dots k$, and each asset a^i is an $m \times n$ matrix, $a^i = \{(a^i)^j_s\}$, where s varies over the states $s = 1 \dots n$, and j varies over the commodities, $j = 1 \dots m$. Let $p \in R^{m\times n}$ be a vector of second-period prices, Δ_2 be the space of all second-period prices, i.e. the strictly positive vectors in the unit simplex in $R^{m\times n}$, and let $p \,\square\, a$ be the matrix of asset returns.[12]

Given two topological spaces, X and Y, we say that X is a *covering space* of Y if there exists an onto map $\theta: X \to Y$ such that, for each $y \in Y$, there is a neighborhood U_y in which the inverse image $\theta^{-1}(U_y)$ is the disjoint union of sets in X each of which is homeomorphic to U_y. For example, the following map θ from the line to the circle, $\theta: R \to S^1$, $\theta(r) = \mathrm{e}^{i\pi r}$ is a covering map which makes the line R^1 a covering space of the circle. When the inverse image of each point y has exactly $k \geq 1$ points, then the covering is called a *k-fold* covering. We denote by $G^{k,n}$ the classic manifold of all k-dimensional subspaces of R^n with the obvious topology, called a *Grassmanian*. Its manifold structure is described in the appendix. We denote by $M^{k,n}$ the space of oriented k-subspaces of R^n; this space differs from the Grassmanian $G^{k,n}$ only in the choice of orientation of the coordinates of the subspaces; this differentiates all changes of coordinates with determinant equal to 1, from those with determinant equal to -1 (see Steenrod, 1951, p. 35). It is immediate that the natural projection $M^{k,n} \to G^{k,n}$ is a *2-fold covering* (see Steenrod, 1951, p. 35, and Singer and Thorpe, 1967, p. 62).

A space X is called a *strong deformation retract* of another Y if $X \subset Y$ and there exists a continuous map $F: Y \times [0, 1] \to Y$ s.t. $\forall y \in Y$, $F(y, 0) = y$ and $F(y, 1) = x \in X$, and $F(x, t) = x$ for all $x \in X$, and all t. If X is a strong deformation retract of Y, then we say that X and Y are *topologically equivalent*, denoted $X \cong Y$. In particular, when $X \cong Y$, they have the same homotopy groups $\pi_i(X) = \pi_i(Y)$ for all $i \geq 1$ (Spanier, 1966).

4.2. The space of real asset matrices $A^{k\times m\times n}$

Since many asset matrices $a \in R^{k\times m\times n}$ yield the same asset returns $\{q^1, \dots, q^k\} \subset R^{k\times n}$, from an economic point of view there is substantial duplica-

[12] Let $p = p(1), \dots, p(n)$, where $\forall s$, $p(s) = (p(s)_{j=1\dots m}) \in R^m$. We define $p \,\square\, a = (q^1 \dots q^k) \in R^{k\times n}$, where $\forall i = 1 \dots k$, $q^i = (q^i)_{s=1\dots n} \in R^n$, $(q^i)_s = \Sigma^m_{j=1}(p(s)_j)\cdot(a^i)^j_s$. For each $p \in \Delta_2$, the map '$\square$' maps an asset matrix $a \in R^{k\times m\times n}$ to a matrix of asset returns, $(q^1 \dots q^k) \in R^{k\times n}$.

tion in representing the space of asset matrices as $R^{k\times m\times n}$. To parameterize the space of pseudo equilibria, it is useful to have a minimal representation for the space $A^{k\times m\times n}$ of asset matrices. We proceed in two steps. In the first we construct a space that contains all the relevant information about real assets, and in the second step we find a unique description of a real asset within this space.

First consider a subspace $S^{k\times m\times n} = \{a \in R^{k\times m\times n} : a = (a^1 \ldots a^k), (a^i)_s^j \neq 0 \Leftrightarrow j = s\}$. Note that $S^{k\times m\times n}$ can be identified with either $R^{k\times n}$ or $R^{k\times m}$. The following lemma shows that the set of asset returns that arise from asset matrices in $R^{k\times m\times n}$ via the map $\square$, is the same as the set that arises from applying the map $\square$ to matrices in $S^{k\times m\times n}$. In other words, there is no loss of information from restricting our attention to the space of asset matrices $S^{k\times m\times n}$.

Lemma 2. Consider any asset matrix $a \in R^{k\times m\times n}$. Then $\forall p \in \Delta_2$ with the first n components of p non-zero there exists an asset matrix $b \in S^{k\times m\times n}$ such that $p \square a = p \square b$.

Proof. Consider first the case in which $m \geq n$, i.e. there are at least as many goods as states. Let $p = p(1), \ldots, p(n) \in \Delta_2$, where $\forall s = 1, \ldots, n$, $p(s) = (p(s)_{j=1\ldots m}) \in R^m$. Now define k $m \times n$ matrices $b = (b^1, \ldots, b^k)$ by $(b^i) = (b^i)_s^j$, $j = 1, \ldots, m$, $s = 1, \ldots, n$, where

$$\text{if } j = s, (b^i)_s^j = (p(s)_j)^{-1}\left\{\sum_{r=1}^{m} (p(s)_r)\cdot(a^i)_s^r\right\},$$

$$(b^i)_s^j = 0 \text{ otherwise.}$$

Then $a \square p = b \square p$. Now consider the case in which $m < n$. A similar construction works if we partition the matrix $(a^i)_s^r$, $s = 1, \ldots, n$ & $r = 1, \ldots, m$ into k $m \times n$ submatrices. $\square$

We have shown in Lemma 2 that it suffices to restrict our attention to asset matrices in S; however, within $S^{k\times m\times n}$ there is still considerable duplication, since two matrices $a = (a^1 \ldots a^k)$, $b = (b^1, \ldots, b^k) \in S^{k\times m\times n}$ define the same asset returns space $L \in G^{k,n}$ for every $p \in \Delta_2$ when the set of vectors $\{a^i\}_{i=1\ldots k}$ and $\{b^i\}_{i=1\ldots k}$ have the same span within the space R^n, and therefore, in practical terms, the two asset structures a and b are the same. For this reason, to obtain a minimal representation for the space of asset returns, it suffices to restrict our attention to the space of all k subspaces of R^n, which is $G^{k,n}$. Therefore in what follows we work with the space of asset matrices $A^{k\times m\times n} = G^{k,n} \subset R^{k\times n}$. We use for $A^{k\times m\times n}$ the same system of charts $\{W_\sigma, \varphi_\sigma\}$ for $G^{k,n}$ as defined in the appendix, but use the notation $A^{k\times m\times n}$ for asset matrices to differentiate it from the space of asset returns.

4.3. The pseudo-equilibrium manifold

The *pseudo-equilibrium manifold* is defined by $\Omega = \{(p, L, w, a) \in \Delta \times G^{k,n} \times R_{++}^{r \times J} \times A^{k \times m \times n}$; (p, L) is a pseudo-equilibrium for the economy with endowments and asset structure $(w, a)\}$; under appropriate regularity assumptions that we adopt here, Ω is a manifold (Duffie and Shafer, 1985).

For each initial endowment vector $w = (w_1, \ldots, w_J)$, $w_i \in R^r$, spot price vector $p \in \Delta$, asset matrix $a \in A^{k \times m \times n}$ and asset return space $L \subset R^n$, denote by $z_i(p, L, w_i, a) \in R^{m \times (n+1)}$ the *demand of agent i* with endowments w_i, facing prices p, asset returns L and asset structure a. There are J traders. For fixed (a, w), summing over all agents i we obtain $\sum_{i=1}^{J}[z_i(p, L, w, a) - w_i] = Z(p, L)$; this is the *excess demand* of an economy with initial endowments $w_1, \ldots, w_J$ and with asset structure a, which we assume to be C^2, i.e. twice continuously differentiable.

4.4. The structure of the pseudo-equilibrium manifold

Consider now an incomplete market economy E as defined in Subsection 2.2:

Theorem 2. Under conditions (i)–(iv), the pseudo-equilibrium manifold Ω is a covering space for a fiber bundle over the Grassmanian manifold $G^{k,n}$. The manifold Ω is topologically equivalent to either the Grassmanian manifold $G^{k,n}$ or to the manifold of oriented k-subspaces in R^n, $M^{k,n}$. In general the pseudo-equilibrium manifold Ω is not contractible when the market is incomplete, i.e. when $k < n$ and $n > 2$. However, when the market is complete, i.e. $k = n$, then the pseudo-equilibrium manifold is $\Omega \equiv G^{n,n} = M^{n,n} = \{R^n\}$, and therefore contractible.

Proof. The proof follows a simple and general line, which can be described informally as follows: we consider a smooth map $F: X \times Y \to Z$ which defines a manifold by $F(x, y) = 0$. For every x, y we show that the Jacobian JF_x with respect to x is a locally constant square matrix. Then we show that this implies that the manifold defined by $F(x, y) = 0$ is a covering space of Y.

To apply this reasoning to our case, the proof has three steps. Step one is to define the map F; F is defined first locally, for each chart system of the manifold $G^{k,n}$, and then globally. Step two then shows that the appropriate Jacobian is locally constant. The final step is to show that the covering covers a fiber bundle over the Grassmanian $G^{k,n}$ or over the space of oriented planes $M^{k,n}$.

The first step is to construct a smooth map H which assigns to each (p, L, w, a), the excess demand vector of the economy, $Z(p, L, w)$, and a k-subspace of R^n. This is done first locally: for each coordinate chart $(W_\sigma, \varphi_\sigma)$ of the manifold $G^{k,n} \times A^{k \times m \times n}$, we define $H_\sigma : \Delta \times W_\sigma \times R^{r \times J} \times A^{k \times m \times n} \to R^r \times R^{k \times (n-k)}$ by

$$H_\sigma(p, L, w, a) = (Z(p, L, w), K_\sigma(p, L, a)), \tag{2}$$

where Z is the excess demand of the economy, and where the map

$$K_\sigma : (\Delta \times W_\sigma \times R^{k \times (n-k)}) \to R^{k \times (n-k)}$$

is defined by

$$K_\sigma(p, L, a) = [I \mid \varphi_\sigma(L)] P_\sigma(p_2 \square a),$$

$[I \mid \varphi_\sigma(L)] P_\sigma$ being the coordinate representation of L in the coordinate chart $(W_\sigma, \varphi_\sigma)$.

The map H_σ has 0 as a regular value (Duffie and Shafer, 1985, Section 6, fact 8, p. 295). The zeros of H_σ define the manifold of pseudo-equilibria Ω; this is because the zeros of any two maps H_σ and $H_{\sigma'}$, are the same on the intersection of each two charts W_σ and $W_{\sigma'}$, so that Ω is indeed a manifold [13] (Duffie and Shafer, 1985, fact 4, p. 295).

The second step is to show that in each chart W_σ, for each spot price $p \in \Delta$, asset return matrix $L \in W_\sigma$ and vector of endowments for all but the first trader $w^{-1} \in R^{r \times (J-1)}$, namely for each (p, L, w^{-1}), there exists a globally invertible map $\Theta_{\sigma,(p,L,w^{-1})} : R^r \times R^{k \times (n-k)} \to R^r \times R^{k \times (n-k)}$ defined by

$$\Theta_{\sigma,(p,L,w^{-1})}(w^1, a) = H_\sigma(p, L, w^{-1}, w^1, a),$$

with a continuous inverse. [14] This second step follows from the fact that, for each (p, L, w^{-1}), the corresponding Jacobian of $\Theta_{\sigma,(p,L,w^{-1})}$ with respect to the variables (w^1, a) is a constant matrix of rank $r + (n-k) \times k$ (see Duffie and Shafer, 1985, p. 295, proof of fact 8). The map $\Theta_{\sigma,(p,L,w^{-1})}$ is therefore globally invertible, and it defines, in particular, a unique inverse for $\{0\} \in R^r \times R^{k \times (n-k)}$, i.e. a unique endowment vector w^1 and a unique asset matrix a in $A^{k \times m \times n}$, denoted $(w^1_\sigma, a_\sigma) = \Lambda_\sigma(p, L, w^{-1})$, which vary continuously with (p, L, w^{-1}), and satisfy $H_\sigma(p, L, w^{-1}, w^1_\sigma, a_\sigma) = 0$, or, equivalently, $(p, L, w^{-1}, w_\sigma, a_\sigma) \in \Omega$.

We have therefore constructed a continuous function $\Lambda_\sigma : [\Delta \times R^{k \times (n-k)} \times R^{r \times (J-1)}] \to R^r \times R^{k \times (n-k)}$, where $\Lambda_\sigma(p, L, w^{-1}) = (w^1_\sigma, a_\sigma)$, and where $H_\sigma(p, L, w^{-1}, \Lambda_\sigma(p, L, w^{-1})) = 0$. The existence of the map Λ_σ implies that for any $L \in W_\sigma$, each element $(p, L, w, a) \in \Omega$ can be represented uniquely as $(p, L, w^{-1}, \Lambda_\sigma(p, L, w^{-1}))$, i.e. the manifold defined by $\Omega_\sigma = \Omega \cap \{\Delta \times W_\sigma \times R^{r \times J} \times A^{k \times m \times n}\}$ can be parameterized by $[\Delta \times R^{k \times (n-k)} \times R^{r(J-1)}]$. In other

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[13] The zeros of the map H_σ are the zeros of the excess demand vector Z, and the subspaces in W_σ which contain $p_2 \square a$, where p_2 is the second-period prices of the vector $p \in \Delta$. The latter property is satisfied because a subspace L in W_σ contains $p_2 \square a$ if and only $K_\sigma(p, L, a) = 0$, since $[I \mid \varphi_\sigma(L)] P_\sigma$ is the representation in the coordinate chart $(W_\sigma, \varphi_\sigma)$ of the space orthogonal to L.

[14] Here we use the notation $w \doteq (w^{-1}, w^1)$.

words, there is a map $\Theta_\sigma : \Omega_\sigma \to [\Delta \times R^{k \times (n-k)} \times R^{r \times (J-1)}]$ defined by $\Theta_\sigma(p, L, w, a) = \Theta_\sigma(p, L, w^{-1}, \Lambda_\sigma(p, L, w^{-1})) = (p, L, w^{-1})$ which is one to one, onto and has a continuous inverse. This completes the second step.

The final step is to show the covering structure. As already noted the zeros of any two maps H_σ and $H_{\sigma'}$ are the same on the intersection of any two charts W_σ and $W_{\sigma'}$, so that by the definition of Λ_σ, for every two coordinate charts σ and σ' of $G^{k,n}$, if $L \in W_\sigma \cap W_{\sigma'}$, then

$$\Lambda_\sigma(p, L, w^{-1}) = \Lambda_{\sigma'}(p, L, w^{-1}).$$

The collection of maps $\{\Lambda_\sigma\}_{\sigma \in \Sigma}$ therefore defines a smooth map $\Lambda : [\Delta \times G^{k,n} \times R^{r \times (J-1)}] \to R^r \times G^{k,n}$, and the corresponding collection $\{\Theta_\sigma\}_{\sigma \in \Sigma}$ defines a map $\Theta : \Omega \to \Delta \times G^{k,n} \times R^{r \times (J-1)}$. The map Θ is locally invertible because $\forall \sigma$, $\Theta / \Omega_\sigma = \Theta_\sigma$, and, as we saw, $\Theta_\sigma : \Omega_\sigma \to [\Delta \times R^{k \times (n-k)} \times R^{r \times (J-1)}]$ is one to one and onto. Since for every chart indexed σ the map Θ_σ is a homeomorphism, it follows that for each neighborhood $U_{(p,L,w^1)}$ of (p, L, w^{-1}) in $\Delta \times G^{k,n} \times R^{r \times (J-1)}$ there exists at most a finite number of disjoint neighborhoods V_σ in Ω, each V_σ homeomorphic to $(U_{(p,L,w^{-1})})$ under Θ, potentially one such neighborhood V_σ for each coordinate chart of $G^{k,n}$. By definition, this means that $\Theta : \Omega \to \Delta \times G^{k,n} \times R^{r \times (J-1)}$ is a covering map (Singer and Thorpe, 1967, p. 63, section 3.3). But $\Delta \times G^{k,n} \times R^{\times (J-1)}$ is a fiber bundle over $G^{k,n}$ with fiber $\Delta \times R^{r \times (J-1)}$, and is topologically equivalent to the Grassmanian $G^{k,n}$ because both Δ and $R^{r \times (J-1)}$ are contractible. This completes the proof of the first part of the theorem.

Now for all k, $n > 2$ with $n - k > 1$, the first homotopy group of $G^{k,n}$, denoted $\pi_1(G^{k,n})$, is cyclic of order 2, i.e. $\pi_1(G^{k,n}) \simeq Z_2$ (see, for example, Steenrod, 1951, p. 134, 25.8, theorem). Since $\pi_1(G^{k,n}) \simeq Z_2$, $G^{k,n}$ has only two possible covering spaces up to a homeomorphism, namely a one-fold and a two-fold cover, each corresponding to the two subgroups of Z_2, $\{0\}$ and Z_2 itself (see, for example, Singer and Thorpe, 1967, theorem 4, p. 71). Therefore either Ω is topologically equivalent to $G^{k,n}$, or alternatively Ω is the simply-connected covering space of $G^{k,n}$, which is the space $M^{k,n}$ of oriented k-planes in R^n (see, for example, Steenrod, 1951, p. 134 and 7.9).

Example of pseudo-equilibrium manifolds. To show that in general the pseudo-equilibrium manifold Ω is not contractible, we provide examples for any $n > k$ and $k = 1$. By the results of Theorem 2, it suffices to study the topology of the manifold $M^{k,n}$ of oriented k-planes in R^n and of the Grassmanian $G^{k,n}$. Both spaces are connected and the former is a two-fold covering of the latter having as a fiber the 0-sphere (see, for example, Steenrod, 1951, 7.9, p. 35). Since $M^{k,n}$ is a covering of $G^{k,n}$, $\forall j \geqslant 2$, all homotopy groups $\pi_j(M^{k,n}) \simeq \pi_j(G^{k,n})$ (see Spanier, 1966), and therefore it will suffice for our purposes to analyze the topology of $G^{k,n}$.

For $k = 1$, $G^{1,n}$ is the projective $(n-1)$-space P^{n-1}, defined as the quotient space of the $(n-1)$-sphere S^{n-1} obtained by identifying antipodal points (see

Greenberg, 1967, p. 21); S^{n-1} is a two-fold covering space of P^{n-1}. For all $k,n>2$ with $n>k$, $\pi_1(G^{k,n})=Z_2$ (see Steenrod, 1951, 25.8, p. 134). Therefore $G^{k,n}$ is not contractible when $n>2$ and $n>k$. Furthermore, when $1 \leqslant i \leqslant k$

$$\pi_i(M^{k,n}) \cong \pi_{i-1}(O_{n-k})$$

where O_{n-k} is the real orthogonal group of transformations in Euclidean space: we have that $\pi_j(O_{n-k+1})=\pi_j(R_{n-k+1})$ for $i \geq 1$, where R_k is the rotation group of O_k which is generally not contractible (see, for example, Steenrod, 1951, p. 35, 7.9, p. 134, 25.8, and p. 131, lines 11–12). Therefore in general Ω is not contractible. □

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Appendix

A.1. The manifold structure of $G^{k,n}$

We describe briefly the manifold structure of $G^{k,n}$: this notation is used in the proof of Theorem 2. A *manifold structure* for $G^{k,n}$ is defined (see Singer and Thorpe, 1967) by a finite set of coordinate charts $\{W_\sigma\}$ covering $G^{k,n}$, and corresponding coordinate maps $\varphi_\sigma : W_\sigma \to R^{k(n-k)}$, which are diffeomorphisms, where $\sigma \in \Sigma = \{\sigma : \sigma$ is a permutation of $\{1,\dots,n\}\}$. $G^{k,n}$ is compact. In each chart $\{W_\sigma\}$ we normalize the vectors to obtain a unique representation of each space, of the form $L=[I \mid K]P_\sigma$, where K is an $(n-k)\times k$ matrix orthogonal to the space L, I is the $(n-k)\times(n-k)$ identity matrix, and P_σ is an $n \times n$ permutation matrix corresponding to σ.

A.2. Proof that regular economies satisfy condition (iv)

The following two lemmas prove Proposition 2 in Section 2 in the text namely, that a *regular* economy satisfies condition (iv) as defined in Section 2. This result is established in two steps. The first step is a preparatory result. Let $S=\{(p,L) \in \Delta \times G^{k,n} : Z(p,L)=0\}$.

Lemma 3. Assume that the economy E is regular Definition 3 *in Subsection* 3.3). *Then each connected component T of S is a covering space of $G^{k,n}$, and for all $j>1$, $\pi_j(T)=\pi_j(G^{k,n})$.*

Proof. By Proposition 1, the projection map $\Pi_2 : S \to G^{k,n}$, defined as $\Pi_2(p,L) = L$, covers its image. Furthermore, since the economy is regular, the Jacobian JZ

of the excess demand Z with respect to the variable p contains an $(r-1)\times(r-1)$ sub-matrix with non-zero determinant $\forall(p, L)\in S$. Therefore, by the implicit function theorem and the hypothesis on JZ (regular economies), for each L in $G^{k,n}$ there exists an open neighborhood N_L of the point L and disjoint open neighborhoods $\{U_\alpha\}$ in S, such that the restriction of the map Z on each U_α is a homeomorphism $Z/_{U_\alpha}: U_\alpha \to N_L$. Since Π_2 covers $G^{k,n}$, this implies, by definition, that $\Pi_2: T\to G^{k,n}$ defines a covering space for each connected component T of S. In particular, for all $j>1$, $\Pi_j(T)=\Pi_j(G^{k,n})$ (see, for example, Croom, 1978, theorem 6.9, p. 116). □

Lemma 4. Under assumptions (i)–(iii) of Section 2, a regular economy E satisfies condition (iv) of Section 2.

Proof. Let E be a regular economy. By Lemma 3 each connected component T of the manifold S is a covering space of $G^{k,n}$. Under assumptions (i)–(iii), for each fixed L there are an odd number of equilibrium price vectors p such that $Z(p, L)=0$: this is a standard result which follows from the boundary behavior of the excess demand assumption (iii). We first consider the case $k>1$ and $n>2$. As shown in the examples of pseudo-equilibrium manifolds following Theorem 2 of Section 4, for all $k>1$ and $n>2$ the space $G^{k,n}$ has only two connected covering spaces: one is $G^{k,n}$ and the other is $M^{k,n}$, the oriented k-fields of R^n, a one-fold and a two-fold covering, respectively. If all components of S had an even number of folds, then the number of equilibria for each L would be even; therefore there must exist at least one connected component T of S such that the covering $T\to G^{k,n}$ has an odd number of folds. This in turn implies that there must exist a connected component of S which is diffeomorphic to $G^{k,n}$; obviously such a component defines a continuous selection from $G^{k,n}$ to an equilibrium price as required by condition (iv).

Finally consider the case $k=1$ and $n=2$. Then $G^{1,2}$ is the circle; since S is a 1-manifold without a boundary, each component is either the line R or the circle S^1. Therefore some component of S must be S^1, for otherwise there would be an infinite number of equilibria, contradicting Debreu (1982); obviously such a component defines a continuous selection. Therefore (iv) is satisfied. □

A.3. A robust example of regular economies which satisfy condition (iv)

Consider an economy with two periods, n states, m goods in each state, J traders, k assets, $k<n$, and let $r=m\times(n+1)$. We assume that all traders have the following identical utility function: [15]

$$u(x_0, x_1, \ldots x_r) = u(x_0) + \rho\sum_{i=1}^{n}\pi_i u(x_i),$$

[15] We are grateful to Yuqing Zhoo for this example.

where

$$u(x_i) = \sum_{l=1}^{m} \alpha_l \log x_{li},\ x_i = (x_{1i}, \ldots, x_{li}, \ldots, x_{mi})$$

$$0 < \rho \leq 1, \quad 0 < \alpha_l < 1, \quad \sum_{l=1}^{m} \alpha_l = 1 \quad \text{and} \quad \sum_{i=1}^{n} \pi_i = 1.$$

We further assume that the first trader has the following initial endowments:

$$\omega_1 = (a, a, \ldots, a) = a\hat{e}, \quad \hat{e} = (1, 1, \ldots, 1) \in R^r_{++};$$

$a \in R^1_{++}$ is some real number which will be chosen later. Also we assume that the other traders have the following identical endowments:

$$\omega_g = (b, b, \ldots, b) = b\hat{e} \in R^r_{++},\ b > 1;$$

b will be chosen later.

From the specifications of the utility functions it is easy to check that the individual demand functions are

$$z^1(p, \omega_1) = az^1(p, \hat{e}),$$

$$z^g(p, L, \omega_g) = bz^g(p, L, \hat{e}) = bz(p, L, \hat{e}),$$

where

$$z^1(p, \omega_1) = \underset{x \in B(p, \omega_1)}{\arg\ \max}\ u(x),$$

$$B(p, \omega_1) = \{x \in R^n : p(x - \omega_1) = 0\},$$

and where

$$z^g(p, L, \omega_g) = \underset{x \in B(p, L, \omega_g)}{\arg\ \max}\ u(x)$$

$$B(p, L, \omega_g) = \{x \in R^n : p(x - \omega_g) = 0 \text{ and } p \square (x - \omega_g) \in L\}.$$

Thus

$$Z(p, L) = az^1(p, \hat{e}) + (J - 1)bz(p, L, \hat{e}).$$

Now we prove that for each $L \in G^{k,n}$ there exists a unique $\hat{p} \in R^{n-1}_{++}$ (we assume that $\hat{p}_{mn} = 1$) such that $Z(p, L) = 0$. Direct computation shows that the price vector

$$\widehat{p_{l0}} = \frac{\alpha_l}{\alpha_m \pi_n}, \qquad \widehat{p_{li}} = \frac{\alpha_l \pi_i}{\alpha_m \pi_n}, \quad i = 1, \ldots, n,$$

satisfies $Z(\hat{p}, L) = 0$ for all L, a and b. Thus the no trade allocation is optimal for each agent and is supported by this price vector. It follows that the equilibrium price is unique.

Let $Z(\hat{p})$ be the first $(n-1)$ coordinates of $Z(p, L)$. $\overline{F}$ and $\overline{p}$ are interpreted similarly. We can readily verify that the form of the utility functions implies that the Jacobian of the first agent's demand function, $Dz^1(p, \hat{e})/Dp$, has a non-zero determinant. We can also compute $Dz(p, L, \hat{e})/Dp$: this is independent of b. It then follows that

$$\frac{D\overline{Z}}{Dp} = a\frac{D\bar{z}^1(p, \hat{e})}{Dp} + (J-1)b\frac{D\bar{z}(p, L, \hat{e})}{Dp}.$$

Since $G^{k,n}$ is a compact manifold, the term $D\bar{z}(p, L, \hat{e})/Dp$ is bounded. Thus when a is sufficiently large and b sufficiently small, it follows that

$$\det\left|\frac{D\overline{Z}(\hat{p}, L)}{Dp}\right| \neq 0, \quad \forall L \in G^{k,n},$$

as we wished to prove.

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