Limited arbitrage is necessary and sufficient for the non-emptiness of the core[1]

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Abstract

A close connection is drawn between the non-emptiness of the core and limited arbitrage, a condition originally defined on the preferences and endowments of the traders of an Arrow–Debreu economy in Chichilnisky (1992, 1993b, 1994, 1995a,b). This condition limits the traders' diversity, and bounds their mutually beneficial gains from trade.

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Editor's Note

This is a revised version of the author's paper "Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and the core and limits voting cycles". The earlier manuscript was inadvertently printed in the December 1994 issue of *Economics Letters* before the completion of refereeing and was subsequently retracted. The current version omits several results from the earlier manuscript that Professor Chichilnisky has in the meantime published elsewhere.

1. Introduction

The expression *limited arbitrage* is used to describe economies where only bounded, or limited, opportunities for gains are available to the traders at their initial endowments. Originally introduced in Chichilnisky (1992, 1993b, 1994, 1995a,b), the concept is central to

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the problem of resource allocation and is closely associated to the social diversity of the economy (Chichilnisky, 1994). Indeed, limited arbitrage was shown to be necessary and sufficient for the existence of a competitive equilibrium in economies with or without short sales in Chichilnisky (1992, 1993b,c, 1994, 1995a,b), and in economies with infinite dimensional spaces in Chichilnisky and Heal (1991, 1993d).[2] In this paper, it is also shown to be necessary and sufficient for the non-emptiness of the core.

2. Limited arbitrage and gains from trade

To offer a formal perspective one needs a few definitions. An economy E has $H \geq 2$ traders who trade $N \geq 2$ commodities or assets, so that the trading space X is R^N; when short sales are not allowed, the trading space is instead R^{N+}. This paper focuses on the case of markets with short sales, but the results are quite general: for markets with as well as without short sales the reader is referred to Chichilnisky (1992, 1993b, 1994, 1995a,b). A trader i is described by an initial endowment $\Omega_i \in R^N$, and by a preference represented by a utility function $u_i: R^N \to R$, $u_i(0) = 0$, which is concave, increasing, and satisfies mild regularity conditions which include all standard convex preferences used in the literature (see Chichilnisky, 1995a). Everything in this paper is ordinal, namely independent of the utility representation; therefore without loss of generality we may consider utilities where $\sup_{\{x; x \in R^N\}} u_i(x) = \infty$.

One wishes to identify those trading opportunities which could yield unbounded utility increases for the ith trader. These are described by net trades in the set $A_i = \{y \in R^N: \forall x \in R^N, \exists \lambda > 0: u_i(\Omega_i + \lambda y) > u_i(x)\}$, a concept originally introduced in Chichilnisky (1992, 1993b, 1995a,b), which contains global information about the trader. The trader's *global cone*, C_i, is either A_i or its closure $\bar{A}_i$.[3] The trader's *market cone* is the set of all those prices at which all trading opportunities in i's global cone are unaffordable, $D_i = \{p \in R^{N+}: \forall y \in C_i, \langle p, y \rangle > 0\}$.

Definition 1. The market economy E has limited arbitrage when all its market cones intersect: $\cap_{i=1}^{H} D_i \neq \emptyset$.

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. The concept of limited arbitrage can also be interpreted in terms of gains from trade (Chichilnisky, 1995b), defined as the maximum increment in the sum of utilities which the traders can achieve by reallocating the economy's resources among themselves:

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[2] See also Werner (1987). After obtaining the results of this paper, I received a paper by Page and Wooders (1994) on a similar topic.

[3] The *global cone* is A_i when the set of gradient directions to the indifference surfaces of u_i is closed for all i, and it is $\bar{A}_i$, the closure of A_i, when all the indifferences of u_i contain no half lines for all i. The preferences in this paper satisfy one of these two conditions (cf. Chichilnisky, 1995a).

$$\text{gains from trade} = G(E) = \sup\left(\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i)\right),$$

where for all i, $u_i(x_i) \geq u_i(\Omega_i)$ and $\sum_{i=1}^{H}(x_i - \Omega_i) = 0$.

Theorem 1. Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales.

For a proof, see Chichilnisky (1992, 1995a,b). Intuitively this is reasonable: an economy such as that in Fig. 1, where traders wish to take unboundedly large and opposed trading positions, cannot reach an equilibrium. Desired trades are just too diverse to be accommodated within the same economy.

The boundedness of mutually beneficial gains from trade is also fundamental for the existence of the core. Formally, when $X = R^N$:
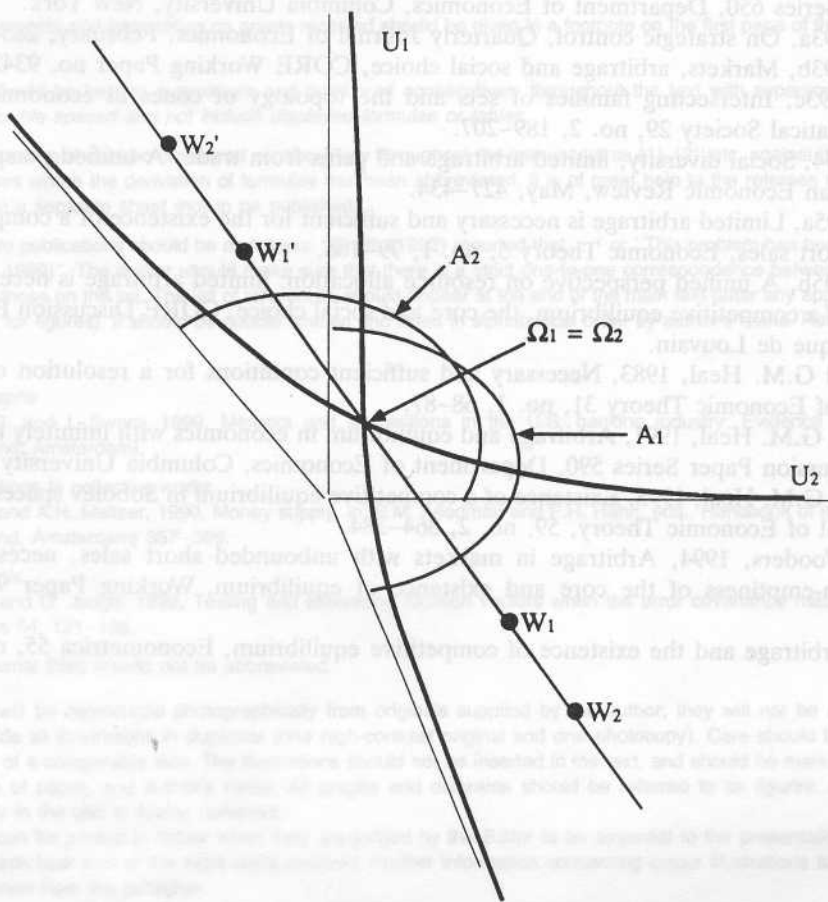

Fig. 1. Limited arbitrage does not hold. The global cones are not contained in a half-space, and there are sequences of feasible allocations such as W_1 and W_1', W_2 and W_2', which produce unbounded utilities.

Theorem 2. Limited arbitrage is necessary and sufficient for the non-emptiness of the core in Arrow Debreu economies.

Proof of Theorem 2. Let $X = R^N$. Since a competitive equilibrium is in the core sufficiency is immediate from Theorem 1. Reciprocally: a core allocation is Pareto efficient, and is therefore a competitive equilibrium for some initial endowments. Since limited arbitrage is satisfied simultaneously at all initial endowments when $X = R^N$, Theorem 1 establishes necessity. □

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