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A robust theory of resource allocation

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Abstract. The theory of social choice introduced in [5, 6] is robust: it is completely independent of the choice of topology on spaces of preferences. This theory has been fruitful in linking diverse forms of resource allocation: it has been shown [17] that contractibility is necessary and sufficient for solving the social choice paradox; this condition is equivalent [11] to another – limited arbitrage – which is necessary and sufficient for the existence of a competitive equilibrium and the core of an economy [13, 14, 15, 16, 17]. The space of monotone preferences is contractible; as shown already in [6, 17] such spaces admit social choice rules. However, monotone preferences are of little interest in social choice theory because the essence of the social choice problem, such as Condorcet triples, rules out monotonicity.

1. Robust resource allocation

The problem of social choice introduced in [5, 6] is the subject of a recent paper by Allen [1]. I welcome Allen's interest, as it indicates that the area is growing and attracting new researchers. Her paper is restricted to a rather special domain, *monotone preferences*, and assumes a special topology, the *closed convergence topology*. Allen states that social choice rules which satisfy my three axioms – continuity, anonymity and respect of unanimity – exist under these special conditions. From this she argues that my social choice paradox is easily resolved, and that it depends on the topology one chooses for space of preferences. In response, this paper will establish that:

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(1) Monotone preferences are of little interest in social choice because they assume that we all share the same view of what allocations are better: this is an implausible situation. The interesting classical problems of social choice arise precisely when people have different opinions. The essence of the social choice problem is the Condorcet triple, and as shown below non-monotonicity is at the heart of this. Allen's framework is trivializing the problem by ruling out from the beginning what many people consider to be the core of the social choice problem. Looking for solutions in spaces of monotone preferences where everyone agrees on what is better, does not give a good indication of the value of the theory.

(2) The existence of social choice rules for spaces of monotone preferences is not a new result: this has been well-known for many years. Indeed, this existence result has appeared in print in my own work starting about 13 years ago, in [6] and [17]. Allen fails to give attribution to these earlier results.

(3) *My social choice paradox is robust: it is completely independent of the choice of topology* on spaces of preferences. Therefore Allen's statement that my paradox depends on the topology is not correct. Indeed, for *any* topology used my social choice paradox holds if and only if the space of preferences is not contractible[1]: this is Theorem 1 in [17]. The space of monotone preferences is contractible under most topologies. Therefore spaces of monotone preferences admit social choice rules. This is immediate. It is also not very interesting for the reasons stated above.

(4) In addition to being robust, the theory of social choice introduced in [5, 6] has led to unexpected and interesting connections with several other forms of resource allocation. The same condition – contractibility – which is necessary and sufficient for solving the social choice paradox [17] has been shown recently [11] to be equivalent to another condition–limited arbitrage – which is necessary and sufficient for the existence of a competitive equilibrium and the core [13, 14, 15, 16].

In brief: My social choice theory is robust, namely independent of the topology chosen for spaces of preferences. It has led to a complete characterization of the spaces where there is a social choice function, and to useful connections with several other forms of resource allocation. Allen's paper [1] is simply a restatement of some of my earlier results [6, 17]. The reason Allen's social choice problem appears to be too simple, is that her formulation trivializes it. Her restriction to the domain of monotone preferences, which as shown below is unnatural in social choice theory, assumes away all the interesting features of the problem.

2. My Paradox holds for any topology one chooses for spaces of preferences

In 1980 I introduced three axioms on aggregation maps from individual to social preferences: continuity, anonymity and respect of unanimity, and proved that such maps do not exist for general spaces of preferences on euclidean space [4–6]. In addition I showed that it *is* possible to construct social choice rules provided one considers restricted preference domains. Indeed, I gave a necessary and sufficient condition on preference domains for the existence of social choice rules: that

[1] If the space is not connected, the precise statement is that the social choice paradox is resolved if and only if every connected component is contractible, see [17].

the spaces of preferences be contractible[2], Chichilnisky and Heal [17]. This latter result [17] is valid for *any* topology on spaces of preferences[3].

It may be useful to explain why the result in [17], which is of a topological nature, is nevertheless quite independent of the topology chosen. Perhaps the best explanation is by way of an analogy with classical topological results used routinely in economics.

Example 1. (Existence of Optima). *A classical theorem of calculus states that a continuous function $f: X \to R$ on a compact space achieves a maximum. This theorem uses the topology of the space heavily, since, without compactness and continuity, two important topological properties, the result is false. Yet the theorem itself does not depend on the topology used on X: no matter what topology is used on the space X, if the space X is compact and the function f is continuous in the chosen topology, a maximum exists.*

Example 2. (Fixed point theorem). *Another example is the classical fixed point theorem used routinely to prove existence of market equilibria. If a space X is compact and contractible[4] and a function $f: X \to X$ is continuous, then there exists a fixed point, $x \in X$ such that $f(x) = x$. Again, it does not matter what topology is used on X: as long as the space X is compact and contractible and the function is continuous in the chosen topology, the fixed point exists.*

The parallel theorem for my axioms of social choice is:

Theorem 3. (Social choice paradox). *A space of preferences P admits a continuous, anonymous social choice rule $\Phi: P^k \to P$ which respect unanimity for every $k > 0$ if and only if every connected component of P is contractible. This result is independent of the topology chosen for the space of preferences P*[5].

For a proof see [17].

The choice of topology does not change the results on fixed point theorems, or the results on the existence of an optimum, or my social choice theorem. Of course a function may fail to be continuous in one topology even if it is continuous in another. Equally, a space may fail to be compact or contractible under one topology even if it is compact and contractible under another. All these qualifications apply to the optimum theorem, to the fixed point theorem and to my theorem. Observe, however, that the fact that continuity changes with the choice of topology has never been used to argue against the usefulness or the importance of the optimum theorem, or that of the fixed point theorem. The usefulness of these

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[2] A topological space X is contractible if it can be continuously deformed through itself into one of its points: there exists a continuous map $F: X \times [0, 1] \to X$ and $x_0 \in X$, such that $\forall x \in X: F(0, x) = x$, and $F(1, x) = x_0$. The results in [5] and [17] established the topological foundations of social choice theory.

[3] The topological space must have minimal regularity conditions such as e.g. locally convexity. If the space is not connected, the theorem requires that every connected component be contractible.

[4] Compactness is not needed: all that is required is that the space X be contractible, provided it is a locally convex space. A space X is contractible when there exists a map $f: X \times [0, 1] \to X$ such that $\forall x \in X, f(x, 0) = x$ and $f(x, 1) = x^0$, for some $x^0 \in X$.

[5] The "universe" for the space of preferences P contains finite or infinite dimensional spaces, manifolds or polyhedra; it suffices that P is a locally convex space. One can give a larger universe for P, by requiring that P be a paracompact CW complex.

theorems is that they tell us *a priori* when the problem has a solution: if we find a topology when the set is compact and the function is continuous, we know for a fact, before making any computation, that an optimum exists. The same holds for the existence of fixed points and of social choice rules. These three theorems, the optimum theorem, the fixed point theorem, and the resolution of the social choice paradox theorem, are "birds of a feather", a point which was established rigorously in [4] and [11].

To emphasize the fact that my paradox does not depend on the topology chosen for spaces of preferences, I covered in my work several possible topologies on spaces of preferences: smooth norms [5], the euclidean metric on spaces of linear preferences [6], the Hausdorff and the related closed convergence topology [10], and my own "order" topology on families of sets introduced in [7]. I considered cases where the space of preferences was connected [5] as well as cases where it is disconnected as in [6, 8] and [17], cases where the preferences admit satiation [6] and [8], and others where they do not [5]. Under some of these topologies the spaces of preferences are manifolds [5, 6], and under others they are not [10, 8]; in any case *my results hold equally for any topology which one chooses on the spaces of preferences.*

My axioms for social choice are somewhat different from Arrow's, although Baryshnikov [2] has recently established an equivalence between the two sets of axioms. My axioms are perhaps better suited than Arrow's axioms for euclidean choice spaces: using these axioms unexpected and intriguing links have emerged recently between social choice, markets, and games [12, 13, 14, 15, 16], the latter two fields typically involving euclidean frameworks.

3. Monotonicity

Allen's paper [1] reconsiders my axioms of social choice within the domain of continuous monotone and complete preorders – a domain which, as stated above, I already considered earlier [3, 6, 9, 17]. She endows preferences with the closed convergence topology, a topology I used before in [10]. Within this framework she repeats an observation which I also made earlier, see e.g. [6, 17]: that over this restricted domain my paradox admits a solution, i.e. there exists an aggregation map which is continuous, anonymous and respects unanimity. Allen seems puzzled by her findings, and states: "The nature of my analysis seems to indicate that the above formulation of the social choice problem may be unsatisfactory. It appears to be too easy to find preference aggregation rules which satisfy continuity, unanimity and respect anonymity".

I am myself somewhat puzzled about Allen's paper, because the points made in her paper have been made several times in the economics literature, for example, in my original 1982 paper in the Quarterly Journal of Economics [6, p. 346, paragraph 1, and Fig. VI]. I quote from [6]:

"For example, when individual preferences are all convex and *increasing*, the "averaging rule" that assigns to each couple of individual surfaces corresponding to the different individuals, say I_{p1}, I_{p2}, the "average" of the two, say I_p in Figure VI, satisfies conditions (1), (2) and (3) (my three axioms). More general examples of this type can also be constructed for restricted domains of preferences, see Chichilnisky (1976)"[6].

A similar result appears also in [17].

[6] Parenthesis and quotations provided.

The "averaging" of preferences mentioned in [6, Fig. VI] was introduced in Chichilnisky (1976)[7].

Two clarifying comments are in order. One is that, as already indicated, Allen's results are not new. For monotonic preferences as considered by Allen [1] the existence of social choice rules has been known for many years, cf. [6], Chichilnisky and Heal [17] and Heal [18]. Perhaps Allen thought it was worth considering the case where preferences are not convex and the closed convergence topology: but this topology was already used by me in [10] and in fact she refers to this paper of mine as well, so she knows this is not new. In any case, as already observed, the topology used does not matter for the results. The convexity of preferences is also not essential to the results, a point already made in [3] and [9] and in Heal [18]. There is therefore nothing new in [1].

The second comment goes to the substance of social choice. It indicates the need for a better understanding of the social choice problem, and addresses Allen's comment that "the problem appears to be too simple".

It is generally useful to consider restricted domains of preferences, because this clarifies when the problem has a solution and when it does not. Indeed this was first done by Black, who solved Arrow's paradox by restricting the problem to single peaked preferences, and was also done in [17]. But certain domains make more sense than others. The domain of preferences considered in [1], i.e. monotonic preferences, may be natural in market analysis, but is of little interest in social choice theory. The reason is that while market analysis considers preferences defined over private consumption – where more is generally better – social choice typically considers, instead, preferences over allocations. Preferences over private consumption are naturally monotonic, but preferences over allocations are not. Indeed, there is no natural sense to the word "more" on the space of allocations, since by definition all feasible allocations add up to the same total. There is no natural order on the set of feasible allocations, which is a subset of a slanted plane in R^N. More on this below.

As its name indicates, the problem of social choice is choosing over social matters, such as allocations between agents, and not over private matters, such as private consumption. It is true that preferences over private consumption define preferences over allocations. But the preferences they define over allocations are typically *not monotonic*: the next section gives standard, very simple and well known examples.

The assumption in [1] that the social choice problem should have a domain of preferences which are all monotonic is implausible and, therefore, of little interest.

4. Monotonicity is irrelevant in social choice

It is well known that monotonicity of preferences over allocations is an implausible assumption. This section will develop several standard, well known, examples, which were already provided in Chichilnisky [6] and in Chichilnisky and Heal [17].

[7] Indeed the space of convex preferences is not convex under the addition of the utilities which represent them, so a special averaging procedure was introduced in Chichilnisky (1976), see [3] and [9].

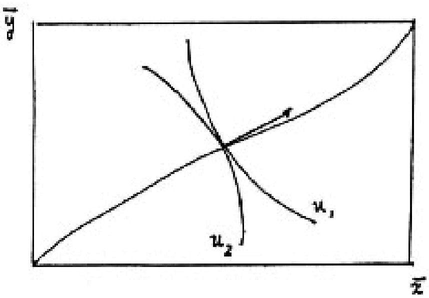

Fig. 1

4.1. Cobb Douglas private preferences lead to non-monotonic preferences over allocations

Monotone preferences are standard in market analysis, so we may take this to be our starting point as it gives the best chance to the contrary position. Consider two traders for whom more private consumption is better. Their consumption space is R^2_+. Assume that each trader has a Cobb Douglas, and therefore monotone, utility over their own private consumption,

$$u_1(x_1, y_1) = x_1^{\alpha} y_1^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

$$u_2(x_2, y_2) = x_2^{\beta} y_2^{1-\beta}, \quad 0 < \beta < 1. \tag{2}$$

Social choice theory considers a problem which is fundamentally different from that of markets: in social choice voters typically choose over all possible allocations of finite amounts of commodities between all individuals in society. A classic example is the Edgeworth box used in microeconomic textbooks. Two traders, 1 and 2, choose over a set of feasible allocations of two finite commodities, x and y. Each individual chooses among the set of all feasible allocations F, consisting of four dimensional vectors, each of which is an allocation of both goods for both people, summing up to the fixed total endowments of the two goods, $\bar{x}$ and $\bar{y}$:

$$F = \{a = (x_1, y_1, x_2, y_2) \geq 0: x_1 + x_2 = \bar{x} \text{ and } y_1 + y_2 = \bar{y}\}. \tag{3}$$

The traders' utilities for private consumption define preferences over allocation as follows: trader i prefers allocation $(x_1, y_1, x_2, y_2) \in F$ to allocation $(x'_1, y'_1, x'_2, y'_2) \in F \Leftrightarrow u_i(x_i, y_i) > u_i(x'_i, y'_i)$.

Now, an allocation can be uniquely identified with a two dimensional vector (x_1, y_1): $0 \leq x_1 \leq \bar{x}$ and $0 \leq y_1 \leq \bar{y}$. It is therefore straightforward to rewrite (1) and (2) taking account of the constraint (3), and obtain

$$u_1(x_1\ y_1) = x_1^{\alpha} y_1^{1-\alpha},$$

$$u_2(x_2\ y_2) = u_2(\bar{x} - x_1, \bar{y} - y_1) = (\bar{x} - x_1)^{\beta}(\bar{y} - y_1)^{1-\beta}.$$

It is immediate that in this standard social choice problem preferences are not monotonic; $\partial/\partial x_1(u_1) > 0$ and $\partial/\partial y_1(u_1) > 0$ while $\partial/\partial x_1(u_2) < 0$ and $\partial/\partial y_1(u_2) < 0$ as Fig. 1 illustrates.

This point was already made in Chichilnisky [6, p. 347, paragraph 1]. In sum: even though preferences over private consumption are monotone, the preferences which they induce over allocations are typically not monotonic. Assuming

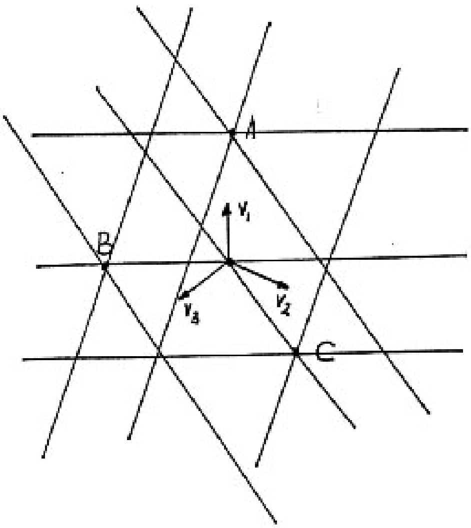

Fig. 2

monotonicity of preferences as in [1] precludes some of the most standard problems of social choice.

4.2. Condorcet triples are not monotone

Another classic problem of social choice was posed by Condorcet several centuries ago. Consider three individuals, 1, 2, and 3, who choose over three choices: A, B, and C. Individual 1 prefers A, B, and C in this order; number 2 prefers C, A, B and number three prefers B, C, A. Condorcet's observation, which is the building block of Arrow's impossibility theorem, was that a majority of two people prefer A to B, another majority prefers B to C, and yet a third majority prefers C to A. Therefore majority voting cannot give rise to a transitive social preference. I will show that this classic example leads to preferences which are naturally not monotonic.

Again by analogy with market analysis, allow traders to choose among all possible vectors in R^2_+ and situate the choices A, B and C in a general position in R^2_+, see Fig. 2 above. Then the existence of a Condorcet triple automatically precludes monotonicity of preferences. For example, when preferences are linear, as Fig. 2 indicates, the individual who prefers A to B and to C must have a gradient vector v_1; individual 2's gradient is v_2 and for individual 3 the corresponding gradient is v_3.

There is no convex cone of directions along which all three individuals are increasing, and therefore no sense to the statement that all three individuals have monotone preferences. This argument and the same diagram were presented in Chichilnisky [6, p. 342, Fig. III]. Indeed, it is intuitively clear that non-monotonicity is at the heart of the Condorcet paradox, and therefore at the heart of the social choice problem.

4.3. Arrow's theorem and monotone preferences

Since one of Arrow's axioms is independence of irrelevant alternatives, his social choice problem can be reduced, as he himself does in his proofs, to three choices (or alternatives) called A, B and C. After reducing the problem to these three choices, Arrow uses Condorcet triples to prove his impossibility theorem. If we assumed that all individual preferences are monotone over triples, then we would rule out Condorcet triples, and we would have ruled out the root of Arrow's paradox by assumption. Requiring monotone preferences would be ruling out interesting and classical examples, the most classical examples in the area. It is not surprising therefore that my paradox disappears when one assumes monotonicity, as I observed in 1982 [6].

4.4. Unrealistic altruism

As already mentioned, assuming that everyone's preferences increase in the same directions is assuming away a standard problem of resource allocation. One could however consider another interpretation of preferences over allocations: rather than being induced by preferences over private consumption as done above, they could be primitively defined over allocations. Each individual could care for what others are assigned, possibly in an altruistic manner – I am better off when you are. However, in this case to assume that everyone's preferences increase in the same direction over allocations is to require a completely unrealistic amount of altruism, implying, for example, that all individuals are better off when one of them is better off, quite independently from their own private consumption. My impression from reading her piece is that altruism is not what Allen had in mind.

5. Strictly convex preferences

Allen [1] refers to a paper by Le Breton and Uriarte [19] where they find a social choice rule satisfying my axioms for spaces of strictly convex preferences with the closed convergence topology. However, Allen fails to mention that Le Breton and Uriarte consider only "single peaked" strictly convex preferences, which are preferences having a unique satiation point. In fact, on the space of all strictly convex preferences my paradox holds, see e.g. Chichilnisky [5, 6]: there are no continuous anonymous rules respecting unanimity, for any number of individuals. A casual reading of Allen's paper appears to indicate the opposite, so it is important to set the record straight.

I have already responded to Le Breton and Uriarte in print, cf. [10], but it may be worth reminding the reader of the response. Le Breton and Uriarte's paper simply repeated a result which was proven earlier by me and Heal in [17]: as I proved in [10] the space of preferences which they consider in [19] is a *contractible* space. It can be deformed continuously into the space of single peaks which is a cube in R^N and therefore contractible. Since my results with Heal [17] established that contractibility is sufficient (as well as necessary) for resolving my social choice paradox, and their space of preferences is contractible, it follows immediately their space of preferences admits a social choice rule satisfying my three axioms[8]. There is no surprise here. Le Breton and Uriarte simply repeated

[8] Note that in [10] I construct a retraction of the space of preferences in [19] into the cube of the preferences' "single peaks"; therefore the space is topologically equivalent to a locally convex space.

a special case of my earlier work [17]. They used a different topology, but as already stated above, the topology does not matter for the results of [17] are valid for all topologies. *For any topology*, and certainly for the closed convergence topology, the space of preferences admits a rule satisfying my axioms if and only if it is contractible.

Le Breton and Uriarte apparently thought that they had found that my results are not robust with respect to different topologies. But this has been corrected in print more than four years ago [10], and Allen knows this, since she refers to my paper where the correction to their error appears. Therefore there is no reason for Allen to repeat an error that has already been corrected.

The error in [19] originated in thinking that their space of preferences was not contractible because at each point in the space all possible gradients can occur. But the structure of the set of gradients at a point is quite irrelevant for the global topology of the space of preferences. For example, the topology on spaces of preferences used in Theorem 1 in Chichilnisky and Heal [17] is not necessarily sensitive to the gradients at each point[9]. In fact, my social choice rules need not be defined on the gradients at each point: My 1982 aggregation example (in Fig. VI, p. 347 of [6], quoted above) constructs a social choice rules on spaces of preferences which disregards completely the gradients at each point. Le Breton and Uriarte's error was corrected in my response more than four years ago, and it suffices to refer the reader to this response [10] where I showed that their space of preferences was contractible, and therefore by my previous theorem [17], it admits a social choice rule. As already mentioned, my results on necessary and sufficient conditions for the existence of social choice rules satisfying my three axioms are completely independent of the choice of topology. They are therefore robust in this sense.

Allen's paper offers a view of the printed record which could confuse the reader. This, plus the lack of originality in her findings, is somewhat puzzling. The puzzle may be more connected with sociological factors than with purely scientific factors: are we perhaps witnessing what Kuhn described as the older generations' resistance to change?[10]

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[9] Indeed the closed convergence topology which they use is insensitive to the gradients at each point.

[10] Kuhn used the terminology "scientific revolutions" to describe scientific change. In this he differed from the everyday use of the word "revolution". He was concerned with the changes in thinking that were required by the introduction of a new idea or theory, even if the idea or the theory were not of the level of importance that one generally associates with the word "revolution".

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