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Economic Theory

Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales*

Graciela Chichilnisky

Departments of Economics and Operations Research, Standord University, 421 Terman, Stanford University, Stanford, CA 94305, and Columbia University, New York, USA

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Summary. A condition of *limited arbitrage* is defined on the endowments and the preferences of the traders in an Arrow-Debreu economy. Theorem 1 establishes that *limited arbitrage* is necessary and sufficient for the existence of a competitive equilibrium in markets with or without short sales. Limited arbitrage bounds utility arbitrages, the diversity of the traders in the economy, and the gains from trade which they can afford from initial endowments (Proposition 2); it is related to but nonetheless different from the no-arbitrage condition used in finance. Theorem 2 establishes that an Arrow – Debreu economy has a competitive equilibrium if and only if every one of its subeconomies with $N + 1$ traders does, where N is the number of commodities. Limited arbitrage has been shown elsewhere to be equivalent to the existence of the core [16], to the contractibility of spaces of preferences and to the existence of continuous anonymous social choice rules which respect unanimity [10], [14], [15], [16].

1. Introduction

A classic problem in economics is how to allocate finite resources among different individuals or groups. Markets provide a widely used solution. Under competitive conditions market clearing allocations are Pareto efficient,[1] and this is the market's main attraction from the point of view of resource allocation.

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[1] See Arrow [1]. The efficiency of the competitive equilibrium does not depend on the concavity of preferences nor on the specification of the consumption or the production sets.

A necessary precondition for using the market solution is the existence of a competitive equilibrium. Arrow and Debreu [3] and McKenzie [24], [26] established sufficient conditions for existence, initiating a large literature dedicated to extending and refining the conditions under which a competitive equilibrium exists.[2] The conditions known for existence are however restrictive: they require for example that all traders should own strictly positive amounts of all goods in the economy, a situation that Arrow and Hahn have described as "unrealistic".[3] Without this or similar conditions an otherwise well-behaved economy[4] may fail to have a competitive equilibrium: a classic example was provided by Arrow and Hahn ([2], Chapter 4, page 80). The problem of non-existence of a competitive equilibrium is pervasive. Despite the fact that market allocations are regarded as a practical solution to the resource allocation problem, many standard economies do not have a competitive equilibrium. Here we argue that the problem arises from the diversity of the traders' endowments and preferences.

A distinguishing feature of this paper is that it obtains a minimal set of conditions on an Arrow-Debreu economy to ensure that a competitive equilibrium exists. This means a condition on endowments and preferences which is both necessary and sufficient for the existence of an Arrow-Debreu equilibrium, the first such condition in the literature. Necessary and sufficient conditions are valuable because they serve to identify, characterize and compute solutions, and to compare these solutions with other forms of resource allocations, for example by social choice and game theory. As an illustration consider the necessary and sufficient ("first order") conditions for partial equilibrium analysis of convex problems. These are one of the most widely used tool in economics: they identify and help compute solutions in the theories of the consumer and of the firm, and in optimal growth theory. A necessary and sufficient condition for market allocations could be equally useful.

A second distinguishing feature of this paper is that we consider economies with or without short sales. In our economies net trades are either bounded below, as they are in a standard Arrow-Debreu specification, or they are not bounded at all. This is a considerable extension from the Arrow-Debreu theory.[5] Our markets include therefore financial markets in which short trades typically occur. We establish a necessary and sufficient condition[6] for the existence of a competitive equilibrium: *limited arbitrage*. This condition is defined in terms of initial endowments and preferences and it has a simple geometric interpretation: it bounds utility arbitrages, the gains from trade, and the diversity of traders in the economy, Proposition 2 and

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[2] Reviewed for example in Arrow and Hahn [2] and more recently in McKenzie [25].

[3] [2], Chapter 4, page 90.

[4] E.g. a standard Arrow-Debreu economy with continuous and concave preferences, with positive endowments and with positive orthants as commodity spaces.

[5] Their formalization of markets assume that the consumption sets of the individuals are bounded below, an assumption motivated by the inability of humans to provide more than a fixed number of hours of labor per day.

[6] Preferences are concave and satisfy minimal regularity conditions, including all widely used preferences such as Cobb-Douglas, CES, linear preferences, which are partly linear, other homothetic preferences, preferences which have indifferences intersecting the boundary of the positive orthant, etc.

Chichilnisky [15]. Limited arbitrage is connected but nonetheless different from the no-arbitrage condition used in finance, Section 3.2.

Somewhat surprisingly, the same condition of limited arbitrage is necessary and sufficient for existence of a market equilibrium with infinitely many commodities, cf. [17], and with or without short sales. (Theorem 1).[7] This is unexpected because the non-existence of a competitive equilibrium appears to be different phenomenon in economies with short sales than in economies without short sales. With short sales, the problem of non existence arises when traders with very different preferences or expectations desire to take unboundedly large positions against each other, positions which cannot be accommodated within the same economy. Instead, without short sales, the problem arises when some traders have zero income. Yet we show that in both cases the source of the problem is the same: the diversity of the traders, which leads to discontinuous demand behavior at the potential market clearing prices, and prevents the existence of a competitive equilibrium. The value of the condition of limited arbitrage is that it ensures that the problem does not arise: with or without short sales it bounds the diversity of traders precisely as needed for a competitive equilibrium to exist. Moreover, Theorem 2 establishes that the economy has limited arbitrage if and only if every subeconomy of $N + 1$ traders does, where N is the number of commodities traded in the market.

Two remarks are in order. One is that it would be possible to use lesser concepts of equilibrium, such as quasiequilibrium and compensated equilibrium, or equilibria where there may be excess supply in the economy. These exist under quite general conditions, but fail to provide Pareto efficient allocations and are therefore less attractive from the point of view of resource allocation.[8] For this reason in this paper we concentrate on competitive equilibrium allocations.

A second remark concerns short sales: it seems important to unify the treatment of markets with and without short sales as done here. Short trading involves contracts to deliver assets in quantities which may exceed initial endowments, and is observed quite generally in the financial markets of the world.[9] Short sales are not allowed in the Arrow-Debreu market; an argument for this presented by Debreu, is that labor is a distinguished commodity which can not be credibly offered beyond a physical limit of 24 hours a day, an argument that applies only to the sale of one person's labor at the time an therefore to a rather "thin" market. Going

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[7] We work within a standard framework where preferences are concave and satisfy regularity conditions. These include just about all concave preferences used in the literature so far.

[8] At a *quasi-equilibrium* or at the related *compensated equilibrium* (see Arrow and Hahn [2]) traders minimize costs rather than maximizing utility. Arrow and Hahn [2] consider also equilibria with excess supply. The allocations emerging from these lesser concepts of equilibrium are not Pareto optimal in general.

[9] This is not true in the Arrow-Debreu theory, which restricts net trades and allows no short sales. It has been argued that such restrictions on trading limit the decentralized nature of the market, and should not be imposed exogenously but should, instead, be derived from the individuals' characteristics and behavior, such as traders' endowments and preferences. For otherwise, the market equilibrium inevitably depends on the chosen bounds, and the solution is achieved by fiat rather than by explained through the trading activity. The literature on financial markets allows any amount of short trading quite generally, examples are Hart [21], Werner [31] and Chichilnisky and Heal [13], [17].

beyond such arguments, it seems artifical to impose exogenously defined limits on trading: such limits clash with the aim of decentralization of the trading activity, because they must be based on the knowledge of what other traders own, which should be private information. In addition, often these limits on trading anticipate by themselves what traders will trade at an equilibrium: for example with linear preferences the equilibrium is located at the boundary described by the chosen bounds. This is not satisfactory: the equilibrium is then defined by fiat, on the basis of the chosen bounds, and not by market behavior. For these reasons, we include here markets with and without short sales: the consumption sets of the traders in our markets are either positive orthants as in the classical theory, or the whole Euclidean space as in financial markets. The latter case thus allows short trading of any magnitude. Limited arbitrage has somewhat different economic interpretations with and without short sales. But in both cases it involves the non-empty intersection of "market cones" defined from the traders' preferences at their initial endowments.

It seems useful to mention another interpretation of limited arbitrage because provides additional motivation and links it with other forms of resource allocation. The non-empty intersection of the cones which defines limited arbitrage is equivalent to a *contractibility* condition on the spaces of preferences: this is a topological condition which ensures that the preferences of all traders can be continuously deformed into one (Chichilnisky [10]). It is therefore a form of similarity of preferences, [15] this time in a topological formulation, as was pointed out in Heal [22]. The connection between non-empty intersection of asymptotic cones, and the contractibility property of spaces of preferences allows one to connect the existence of a competitive equilibrium with the existence of social choice rules (Chichilniksy [14]), because it has been already established that the contractibility of the space of preferences in necessary and sufficient for the existence of social choice rules, Chichilnisky and Heal [13], and with the existence of the core [16]. Limited arbitrage is necessary and sufficient for the existence of the core [16], and is also necessary and sufficient for the existence of continuous anonymous aggregation rules which respect unanimity [7], [14].

1.1. Arbitrage and equilibrium

Welfare economics and finance have each evolved their own equilibrium concepts. In welfare economics, this is the competitive equilibrium: in finance, it is the absence of arbitrage opportunities. These concepts emerged independently and were initially seen as quite distinct.[10]

The absence of arbitrage opportunities (a no-arbitrage condition) is clearly necessary for the existence of a competitive equilibrium. If arbitrage opportunities

[10] The first explicit study of the arbitrage-equilibrium relationship was Kreps [23] developed by Hammond [20], Werner [31], and Nielsen [28]. Green [19] and Grandmont ("Temporary general equilibrium theory", Handbook of Mathematical Economics, 1982) gave necessary and sufficient conditions on expectations and excess demand for existence in Green's temporary equilibrium model; related conditions are in Hart [21].

remained at an equilibrium, then the traders could not be maximizing their utility at the equilibrium allocations. The condition of no-arbitrage is therefore an equilibrium condition, one which must be satisfied at an equilibrium allocation; however it does not help to evaluate the economy's ability to reach a competitive equilibrium, which is the problem we study here, in the sense that only after an equilibrium allocation is found one can verify this condition.

It remains therefore to give conditions on the *primitives* of the economy, such as the traders' endowments and preferences, which are both necessary and sufficient for the existence of a competitive equilibrium, and which hold both for economies with and without bounds on short sales. This is accomplished in this paper: we provide a geometric condition on initial endowments and preferences – limited arbitrage – which is necessary and sufficient for the existence of a competitive equilibrium.[11] Limited arbitrage limits, but does not rule out, arbitrage opportunities; it bounds gains from trade in the economy, Proposition 2 and [15]. We prove that a competitive equilibrium exists if and only if arbitrage opportunities at the initial endowments are, in a precise sense, limited. The equilibrium concepts used in economics and finance are therefore equivalent in the context of limited arbitrage. This equivalence contrasts with a conjecture of Dybvig and Ross [18] to the effect that "absence of arbitrage is more primitive than equilibrium, since only relatively few rational agents are needed to bid away arbitrage opportunities".

2. Definitions and examples

An Arrow-Debreu *market economy* with $H \geq 2$ traders and $N \geq 2$ commodities is defined by $\mathsf{E} = \{X, \Omega_h, \rho_h, h = 1, \ldots, H\}$, where X is the consumption or trading space; X is either the positive orthant R^N_+ or all of the Euclidean space R^N. R^N_{++} denotes the interior of X^N_+. The traders are indexed by $h = 1, \ldots, H$; each has a non-zero initial endowment in $R^N, \Omega_h \geq 0$, where $\Omega = \sum_{h=1}^{H} \Omega_h \gg 0$ is the total endowment of the economy. Some individuals may have zero endowments of some goods. Each individual has a preference ρ_h over private consumption, which is continuous, convex and monotonically increasing: if $x \geq y$ then $x \succeq_{\rho_h} y$. The preferences admit a representation by continuous functions: ρ_h is represented by $u_h: X \to R$. All the assumptions and the results in this paper are *ordinal*, in the sense that they are independent of the utility representations. Therefore we may assume without loss of generality that the preferences satisfy $\sup_{\{x: x \in X\}} u_h(x) = \infty$. We may also consider more general specifications of the consumption set X. For example, we will discuss consumption sets X which are translates of the positive orthant, $X = \{v \in R^N : v \geq w \text{ for some } w \in R^N\}$, and convex sets $X \subset R^N$ which are bounded below and satisfy $x \in X, y \geq x \Rightarrow y \in X$, see Chichilnisky and Heal [13].

[11] Previous results in this direction are in Chichilnisky and Heal [13], who obtained sufficient conditions for existence of a market equilibrium which are related to no-arbitrage. Their conditions are however too strong to be necessary in general; these works are compared with the results of this paper in Section 5.2 below. Chichilnisky and Heal deal with finite and infinite dimensional economies, with or without bounds on short sales. Monotonicity is not needed in the proofs; only non satiation is required, cf. Chichilnisky [16].

Assumption 1. When $X = R^N_+$, we require that if an indifference surface corresponding to a positive consumption bundle x intersects a boundary ray[13] $r \subset \partial X$, all indifference surfaces of bundles preferred to x intersect r. This includes all standard preferences on R^N_+ such as: Cobb-Douglas, CES, preferences with indifference surfaces of positive consumption contained in the interior of R^N_+, linear preferences, piecewise linear preferences, Leontief preferences, preferences with indifference surfaces which intersect the boundary of the positive orthant (Arrow and Hahn [2]) and smooth utilities defined on a neighborhood of X which are transversal to its boundary ∂X, Smale [30].[14]

Assumption 2. When $X = R^N$ the preference ρ_h is represented by a smooth (C^2) utility function[15] $u_h: R^N \to R, \exists\, \varepsilon, K > 0: \| Du_h(x) \| > \varepsilon$ and $\| D^2 u_h(x) \| < K$ for all $x \in R^n$, and $\forall h$: (a) the directions of the gradients of each indifference surface which is not bounded below define a closed set or (b) indifferences contain no halflines.[16] Assumption 2 includes all smooth preferences in R^N having indifference surfaces which are contained in the interior of a translate of the positive orthant, as well as preferences whose indifference surfaces are not contained in the interior of any translation of the positive orthant, such as for example, linear preferences, or preferences which have partially linear indifference surfaces; it includes preferences which are extensions to R^n of Cobb-Douglas or CES utilities defined on a closed subset of the strictly positive orthant, and strictly convex preferences which may or not be transversal to the boundary of the positive orthant.

The *space of allocations* is $X^H = \{(x_1, \ldots, x_H) \in R^{NH}: x_h \in X\}$. The *space of feasible allocations* is $\Upsilon = \{(x_1, \ldots, x_H) \in X^H: \sum_{h=1}^H x_h = \Omega\}$. A $k-$ *trader sub-economy of* E is an economy consisting of a subset of $k \le H$ traders in E, each with the endowments and preferences as in E: $\mathsf{F} = \{X, \rho_h, \Omega_h, h \in J \subset \{1, \ldots, H\},$ *cardinality* $(J) = k\}$. The set of *supports to individually rational efficient resource allocations* of the economy $\mathsf{E} = \{X, \Omega_h, \rho_h, h = 1, \ldots, H\}$ is:

$$S(\mathsf{E}) = \{v \in R^N: \exists\, (x_1, \ldots, x_H) \in \Upsilon \text{ with } x_h \ge_{\rho_h} \Omega_h \ \forall h = 1, \ldots, H, \text{ and } \forall z_h \in X, z_h \ge_{\rho_h} x_h \Rightarrow \langle v, z_h - x_h \rangle \ge 0\}. \quad (1)$$

This is the set of prices which support those feasible allocations which all individuals prefer to their initial endwoments; the allocations are efficient because the vectors $z_h - x_h$ are supported by the same v. An element v of $S(\mathsf{E})$ is called a *support for the allocation* $x = (x_1, \ldots, x_H) \in \Upsilon$. The set of *prices orthogonal to the endowments* is

$$N = \{v \in R^N_+ - \{0\}: \exists h \text{ s.t. } \langle v, \Omega_h \rangle = 0\}. \quad (2)$$

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[13] A boundary ray r in R^N_+ is a set which consists of all the positive multiples of a vector $v \in \partial R^N_+ : r = \{w \in R^N_+ : \exists \lambda > 0 \text{ s.t. } w = \lambda v\}$.

[14] These preferences are quite general; consumption bundles that lie in the intersection of a budget sets and the boundary of the positive orthant, may or may not be indifferent to each other.

[15] It is immediate to extend the results of this paper to preferences over $X = R^N$ which are not smooth, at the cost of more notation. For the case $X = R^N_+$ we require no smoothness.

[16] A set is bounded below if all its elements are larger than a given vector. Case (b) was considered previously by Werner [31] and Chichilnisky [16]; both cases involve the same proofs.

N could be an empty set; it is always empty when $\forall h, \Omega_h \gg 0$. Consider now a utility representation u_h for each preference ρ_h, with $u_h(0) = 0$. The *utility possibility set* of the economy E is the set of all possibility utility values which individuals can obtain from feasible allocations:

$$U(\mathrm{E}) = \{(V_1, \ldots, V_H) \in R^N_+ : V_h = u_h(x_h), \quad \text{where } (x_1, \ldots, x_H) \in \Upsilon \text{ and } \forall h = 1, \ldots, H, u_h : X \to R \text{ represents the preference } \rho_h\}.$$

The *Pareto frontier* of the economy E is the subset of vectors in the utility possibility set which are not dominated in the order of R^H:

$$P(\mathrm{E}) = \{(V_1, \ldots, V_H) \in U(\mathrm{E}) : \sim \exists (W_1, \ldots, W_H) \in U(\mathrm{E}) : \forall h = 1, \ldots, H, W_h \geq V_h \text{ and } W_h > V_h \text{ for some } h \in \{1, \ldots, H\}\}. \tag{3}$$

A *competitive equilibrium* of E consists of a price vector $p^* \in R^N_+$ and a *feasible allocation* $(x_1^*, \ldots, x_H^*) \in \Upsilon$ such that x_h^* optimizes ρ_h over the budget set

$$B_h(p^*) = \{x \in X : \langle x, p^* \rangle = \langle \Omega_h, p^* \rangle\}.$$

A *pseudo-equilibrium* of E consists of a price vector $p^* \in R^N_+$ and a *feasible resource allocation* $(x_1^*, \ldots, x_H^*) \in \Upsilon$ such that $\forall h = 1, \ldots, H$, $y \geq_{\rho_h} x_h^* \Rightarrow \langle p^*, y \rangle \geq \langle p^*, x_h^* \rangle$. This implies that the allocation $(x_1^*, \ldots, x_H^*)$ minimizes costs at p^*. Such an equilibrium is also called a *quasi-equilibrium*. A pseudo-equilibrium need not be a competitive equilibrium, because a cost minimizing allocation may not maximize utility within the corresponding budget set. However, when $\forall h = 1, \ldots, H$, $\langle p^*, \Omega_h \rangle > 0$, then a pseudo equilibrium is also a competitive equilibrium, Arrow and Hahn [2].

2.1. Market cones

The next step is to define a family of cones from the "primitives" of the economy: the endowments and the preferences of the traders. The properties of this family of cones are a topological invariant of the economy and characterize the behavior of the economy: in them lies the answer to questions such as whether the economy has a competitive equilibrium. The cones are slightly different in two cases: when $X = R^N$ and $X = R^{N+}$, which are considered separately.

2.2. Case 1: $X = R^N$

Consider a preference ρ_h in E and an initial endowment vector $\Omega_h \in X$. The *global cone of* $\rho_h \in \mathrm{E}$ *at the initial endowment* Ω_h is: (cf. Chichilnisky [16]) in case (a)

$$A(\rho_h, \Omega_h) = \{y \in R^N : \forall z \in X, \exists \lambda > 0 \; s.t. (\Omega_h + \lambda y) \succ_{\rho_h} z\}, \tag{4}$$

its closure in case (b)

2.2.1. Relationship of $A(\rho_h, \Omega_h)$ with other cones in the literature

The cone $A(\rho_h, \Omega_h)$ is the same for all $\Omega_h \in R^N$; it has global information about the trader, and is new in the literature. It has points in common with Debreu's "asymptotic cone" corresponding to the preferred set of ρ_h at the initial endowment

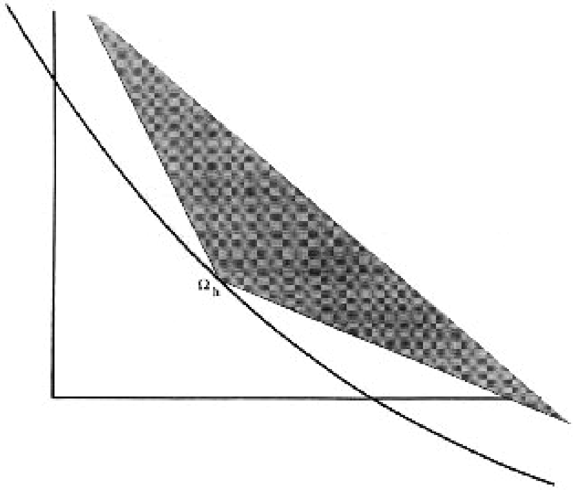

Figure 1. The global cone $A(\rho_h, \Omega_h)$ of a preference ρ_h over $X = R^2$ translated to the endowment Ω_h.

Ω_h, in that along any of the rays of $A(\rho_h, \Omega_h)$ utility always increases, its closure $\bar{A}(\rho_h, \Omega_h)$ is the "recession" cone introduced by Rockafeller by Proposition 1 under Assumption 2, but not generally. However, the similarity with those cones ends here, because along the rays in $A(\rho_h, \Omega_h)$ not only does utility increase forever, but it increases beyond the utility level of any other vector in the consumption space X. In ordinal terms, *the rays of the global cone $A(\rho_h, \Omega_h)$ intersect all indifference surfaces corresponding to bundles preferred by ρ_h to Ω_h.* This condition need not be satisfied by Debreu's asymptotic cones, or by Rockafeller's "recession" cones. Related conditions appear in Chichilnisky [4], [5]; otherwise there is no precedent in the literature for global cones.

Proposition 1. *The global cones $A(\rho_h, \Omega_h)$ of the economy E are open convex sets.*

Proof. Consider a sequence $(v^n)_{n=1,2,\ldots}$, in $C(A(\rho_h, \Omega_h))$, the complement of $A(\rho_h, \Omega_h)$, defining halflines $(\Gamma^n)_{n=1,2,\ldots}$, with (without loss) different $\sup_{\{x:x\in\Gamma^n\}}(u_i(x)) < \infty\ \forall n$. By the assumptions on u_i, $\forall \varepsilon > 0$, $\forall n \exists y \in \Gamma^n$: $\langle Du_h(y), w\rangle \leq \varepsilon$ if $w \in \Gamma^n$. Concavity of u_h implies that $\forall w \in \Gamma^n$, $\langle Du_h(\lambda y), w\rangle \leq \varepsilon \forall \lambda > 1$. Assume that on two halflines $\Gamma^n \neq \Gamma^m$ the utility u_i is eventually constant at $u_h(y_0^m)$ and $u_h(y_0^n)$: $\exists y^n \in \Gamma^n$ and $y^m \in \Gamma^m$ such that as $\lambda \to \infty$ $\langle Du_i(\lambda y^n), w\rangle \to 0 \forall w \in \Gamma^n$, and $\langle Du_h(\lambda y^m), w\rangle \to 0 \forall w \in \Gamma^m$, and $u_h(y_0^n) < u_h(y_0^m)$. Let Π be a supporting hyperplane for the preferred set of u_h at λy^m; this determines a halfspace Λ of R^N: $\forall q \in \Lambda, u_i(q) < u_i(\lambda y^m)$; note that as $\lambda \to \infty$ Π asymptotically contains an unbounded segment of Γ^m, and Λ an unbounded segment of Γ^n. Therefore $\forall K > 0 \exists z^K \in \Gamma^n$ and $w^K \in \Pi: \|z^K - w^K\| > K$ and as $K \to \infty$, $u_h(z^K) \to u_h(y_0^n)$ and $u_h(w^K) \to u_h(y_0^m)$. Since by assumption $\exists \varepsilon > 0: \forall x$, $\|Du_h(x)\| > \varepsilon$, $\forall K$ the distance between z^K and $\{w \in R^N: u_h(w) = u_h(y_0^m)\}$ is bounded: $\exists T > 0: \forall K$, $\|z^K - w^K\| < T$, a contradiction. The contradiction arises from assuming that u_h is eventually constant on Γ^n and Γ^m with $n \neq m$; therefore $\exists n_0: \forall j \geq n_0 \exists y \in \Gamma^j$: $\langle Du_h(\lambda y), w\rangle < 0\ \forall w \in \Gamma^j$ and for λ large. By concavity of u_h, this implies that along the halfline Γ defined by $v = \lim^n v^n$, u_h is bounded, so that $v \in C(A(\rho_h, \Omega_h))$. Thus $C(A(\rho_h, \Omega_h))$ is closed and $A(\rho_h, \Omega_h)$ open. Convexity is immediate. □

The *market cone* of the economy **E** is defined by

$$D(\rho_h, \Omega_h) = \{z \in X : \forall y \text{ in the global cone } \langle z, y \rangle > 0\}. \tag{5}$$

Example 1. *The market cone $D(\rho_h, \Omega_h)$ of a linear preference ρ_h which is defined by its gradient vector, $G \in R^N$, is the vector G itself. The market cone of a preference ρ_h with cone $A(\rho_h, \Omega_h) = R^N_+$, is the same cone, i.e. $D(\rho_h, \Omega_h) = R^N_+$. The market cones of an increasing preference may contain vectors with some negative coordinates, but will not contain strictly negative vectors. In general, the larger is the global cone, the smaller the market cone, and reciprocally.*

2.3. Case 2: $X = R^N_+$

The *global cone* $A(\rho_h, \Omega_h)$ of the *hth* individual in the economy E, is defined as in (4) above:

$$A(\rho_h, \Omega_h) = \{y \in X : \forall z \in X, \exists \lambda > 0 \; s.t. \; (\Omega_h + \lambda y) \succ_{\rho_h} z\}. \tag{6}$$

When $X = R^N_+$ the *market cone* is defined as:

$$\partial D(\rho_h, \Omega_h) = D(\rho_h, \Omega_h) \cap S(E) \text{ if } S(E) \subset N,$$
$$= D(\rho_h, \Omega_h) \text{ otherwise.} \tag{7}$$

where $S(\mathsf{E})$ and N are as defined in (1) and (2) above.

The interpretation of the market cone ∂D_h is as follows. *If all supports in $S(E)$ assign some trader h zero income, then ∂D_h consists of all those supporting prices at which only limited increases in utility can be afforded from intial endowments.*

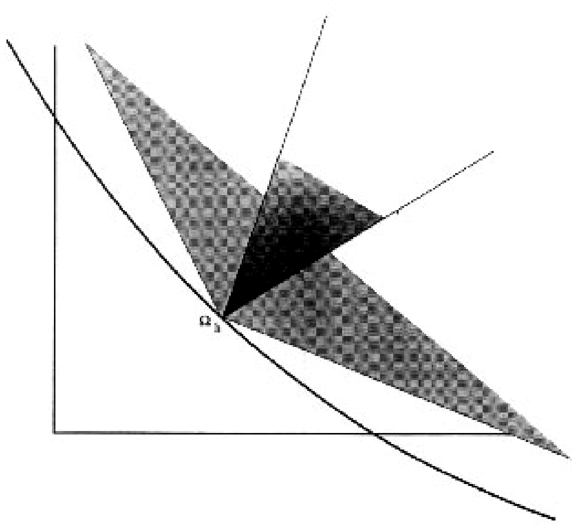

Figure 2. The market cone $D(\rho_h, \Omega_h)$ of the preference ρ_h in Figure 1, translated to the initial endowment Ω_h.

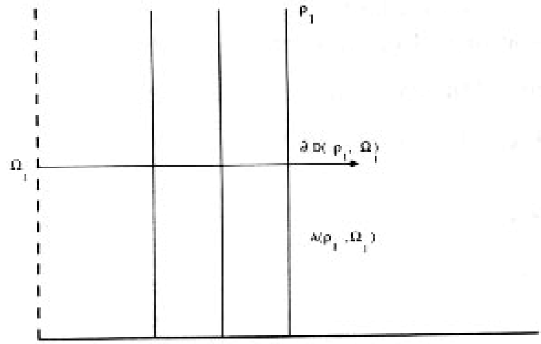

Figure 3. The global cone $A(\rho_1, \Omega_1)$ of the preference ρ_1 is the positive orthant R^2_+ minus the positive part of the vertical axis.

Note that the market cone ∂D_h contains the interior of the consumption set $X = R^N_+$ when $S(E)$ has a support assigning strictly positive income to all individuals. Also, if for some prices some trader has zero income, then this trader must have a boundary endowment.

Figure 4A illustrates the market cone of the preference in Figure 3, in an economy where for all $i = 1, \ldots, H$, the preferences satisfy $\rho_i = \rho_1$ which is indifferent

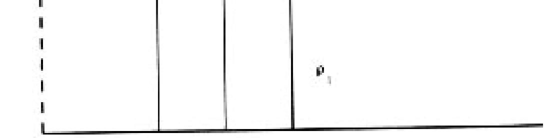

Figure 4A

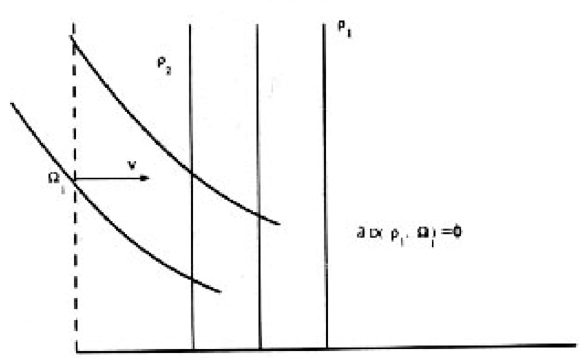

Figure 4B

in the second good, and the endowments are as illustrated. Since Ω_2 is in the interior of X and ρ_2 is indifferent in the second good, then the only possible price in $S(\mathsf{E})$ is the vector v. Since Ω_1 only owns the second good, then for every price in $S(\mathsf{E})$, there exists one individual with zero income, namely trader 1. The global cone $A(\rho_i, \Omega_i)$ is the positive orthant minus the vertical axis of coordinates, for all i. In this economy, for all i the market cone $\partial D(\rho_i, \Omega_i)$ is the half-line spanned by the vector v, because v has strictly positive inner product with all positive vectors including those with second coordinate equal to zero.

Figure 4B illustrates a different economy. It has the same number of traders as the economy in 4a. The endowments and preferences of its traders are the same as those in Figure 4A, except for the preference of trader two, which is now strictly increasing in the second coordinate as illustrated. Here the global cone $A(\rho_2, \Omega_2)$ is the whole positive orthant R^2_+, and the market cone $\partial D(\rho_2, \Omega_2) = \varnothing$, because v is the only vector in $S(\mathsf{E})$ since trader 1 is still present, and $\langle v, y \rangle = 0, \forall y = (y_1, 0) \in A(\rho_2, \Omega_2)$.

Definition 1. *Consider an economy* $\mathsf{E} = \{X \subset, \rho_h, \Omega_h, h = 1, \ldots, H\}$. *When* $X = R^N$ *the* family of market cones of E *is* $\{D(\rho_h, \Omega_h), h = 1, \ldots, H\}$. *When* $X = R^N_+$, *the* family of market cones of E *is* $\{\partial D(\rho_h, \Omega_h), h = 1, \ldots, H\}$.

The market cones $\partial D(\rho_h, \Omega_h)$ depend in general on the initial endowments as well as on the preferences. As the endowment Ω_h varies, the cone $\partial D(\rho_h, \Omega_h)$ may also vary, for example it is always the consumption set $X = R^N_+$ when endowments are strictly interior to X, while it can be empty otherwise, as seen in Figure 4 above. This also differs from the cones used in other works.[17]

3. Limited arbitrage: definition and examples

We consider two cases: Case 1 is when the consumption set is $X = R^N$:there are no bounds on short sales. Case 2 is $X = R^N_+$. The limited arbitrage condition is somewhat different in these two cases, although in both cases it involves the non-empty intersection of market cones. In addition, we discuss the interpretation of limited arbitrage for more general consumption sets and provide a geometric interpretation as a transversality condition.

3.1 Case 1. Limited arbitrage without bounds on short sales, $X = R^N$

Consider a market economy $\mathsf{E} = \{X, \Omega_h, \rho_h, h = 1, \ldots, H\}$, where $X = R^N$ E satisfies *limited arbitrage* if and only if

$$(LA) \bigcap_{h=1}^{H} D(\rho_h, \Omega_h) \neq \varnothing.$$

[17] Such as e.g. in Werner, whose cones are assumed to be the same at all vectors in the consumption set ([31], Assumption A3 and Proposition 1). When $X = R^N$, $D(\rho_h, \Omega_h)$ is the same $\forall \Omega_h \in R^N$ under Assumption 2, but not generally.

3.2. Limited arbitrage and no-arbitrage

In financial markets an *arbitrage opportunity* exists when individuals can make unbounded gains at no cost, or, equivalently, by taking no risks. For example, buying an asset in a market where its price is low while simultaneously selling it at another where its price is high can lead to unbounded gains at no risk to the trader. *No-arbitrage* means that such opportunities do not exist, and it provides a standard way of pricing a financial asset: precisely so that no arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market clearing equilibrium there is no-arbitrage.

The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy E where the traders' initial endowments are zero, $\Omega_h = 0$ for all h. Here *no-arbitrage* at the initial endowments means that there are no trades which could increase the traders' utility at zero cost: gains from trade in E must be zero. By contrast, E has *limited arbitrage* when no trader can increase utility beyond a given bound at zero cost; as seen in Chichilniksy [15] (Proposition 2 of Section 1), gains from trade are bounded. In summary: no-arbitrage requires that there should be *no* gains from trade at zero cost while limited arbitrage requires that there should be only bounded utility arbitrage or *limited* gains from trade.

The two concepts are related but nonetheless quite different. No-arbitrage is a market clearing condition: it is used to describe an allocation at which there is no further reason to trade. It can be applied at the initial allocations, but then it means that there is no reason for trade in the economy as a whole: the economy is autarchic and therefore not very interesting. By contrast, limited arbitrage is applied only to the economy's initial data, the traders' endowments and preferences, and it does not imply that the economy is autarchic. It is valuable in predicting whether the economy can ever reach a competitive equilibrium, and allows us to do this simply by examining the economy's initial conditions.

3.3. Limited arbitrage with bounds on short sales $X = R^N_+$

Consider now a market economy $E = \{X, \Omega_h, \rho_h, h = 1, \ldots, H\}$, where $X = R^N_+$. *Limited arbitrage* is:

$$(\partial LA) \bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) \neq \varnothing \tag{8}$$

where the market cones $\partial D(\rho_h, \Omega_h)$ are defined in Section 2, (7).

This condition ensures that *if all the supporting prices in $S(E)$ assign zero income to some trader, there is one at which only limited (or bounded) increases in utility are affordable from initial endowments.*

Example 2. *Examples of economies which satisfy limited arbitrage and of preferences which do not. When the consumption set is $X = R^N_+$, limited arbitrage is always satisfied if all indifference surfaces through positive consumption bundles are contained in the interior of X, R^N_{++}. Examples of such preferences are those given by Cobb-Douglas utilities or by CES utilities with elasticity of substitution $\sigma < 1$. This*

This condition can be interpreted as follows: *there exists a price p at which only limited (or bounded) increases in utility are affordable from initial endowments for all traders.* In case (a), limited arbitrage can be interpreted as follows: *gains from trade in the economy are bounded.*[18] *Gains from trade* are in the economy E are denoted $G(\mathsf{E})$ and defined as follows:

$$G(\mathsf{E}) = \sup \left\{ \sum_{H=1}^{h} (u_h(x_h) - u_h(\Omega_h)) \right\},$$

where $\sum_{h=1}^{H} (x_h - \Omega_h) = 0$, and $\forall h,\ u_h(x_h) \geq u_h(\Omega_h)$. In case (a):

Proposition 2. *Limited arbitrage is satisfied if and only if*

$$G(\mathsf{E}) < \sum_{h=1}^{H} \left(\sup_{\{x : x \in X\}} u_h(x) - u_h(\Omega_h) \right), \quad or\ G(\mathsf{E}) < \infty,$$

when $\sup_{\{x:x\in X\}} u_h(x) = \infty$.

Proof. A proof is in Chichilnisky [15], [16].

Examples of economies which do not satisfy the limited arbitrage condition when $X = R^N$ are those where the individuals have different linear preferences, Figure 5. In Figure 5 the global cones of the preferences are open half spaces, and the market cones are the two gradient vectors defining the preferences. Clearly, if the preferences are linear and different these market cones do not intersect. In Figure 6 each two market cones intersect, but the three market cones do not intersect, and the economy violates limited arbitrage. This figure illustrates the fact that the union of the market cones may fail to be contractible: indeed, this failure corresponds to the failure of the market cones to intersect, as proven in Chichilnisky [10].

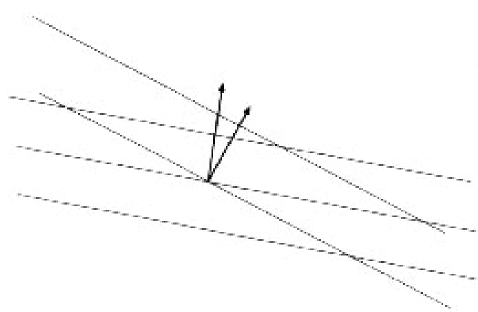

Figure 5. $X = R^2$. Two individuals with different linear preferences. Limited arbitrage fails, and the economy has no competitive equilibrium.

[18] Since all assumptions and results are ordinal and do not depend on the utility representations, without loss of generality we can normalize preferences so that $\sup_{\{x:x\in X\}} u_h(x) = \infty$. If a different normalization is required, it suffices to replace "∞" by "$\sup_{\{x:x\in X\}} u_h(x)$" in all the statements and results.

is because all such preferences have the same global cone, namely the positive orthant, and therefore their market cones always intersect. Since their global cones are identical, these preferences are very similar to each other on choices involving large utility levels. This is a form of similarity of preferences.

Example 3. *An example of an economy with $X = R^N_+$ which does not satisfy limited arbitrage in this case is illustrated in Figure 7. As shown in Figure 4B above, in this economy the dual cone of the first trader is empty, $\partial D(\rho_1, \Omega_1) = \emptyset$, so limited arbitrage as defined in (8) is violated. This economy has no competitive equilibrium.*

Example 4. *Economies where the individuals' initial endowments are strictly interior to the consumption set X always satisfy the limited arbitrage condition in the case $X = R^N_+$, since in this case $\forall h$, $\partial D(\rho_h, \Omega_h) \supset R^N_{++}$ for all $h = 1, \ldots, H$. This is because all individuals have non-zero income at any supporting price in S(E).*

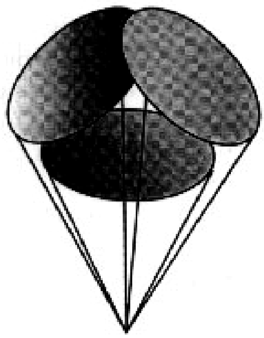

Figure 6. $X = R^3$. Every two trader subeconomy satisfies limited arbitrage, but in the economy as a whole limited artitrage fails. There is no competitive equilibrium.

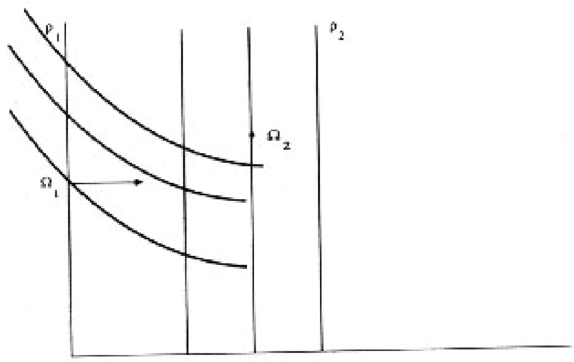

Figure 7. $X = R^2_+$. The market cones do not intersect and limited arbitrage fails. The economy has no competitive equilibrium. This is similar to an example in Arrow and Hahn [2].

When $X = R^N_+$ the limited arbitrage condition may fail to be satisfied when some trader's endowment vector Ω_h is in the boundary of the consumption space, ∂X, and at all supporting prices some trader has zero income: $\forall p \in S(\mathsf{E}) \exists h$ such that $(p, \Omega_h) = 0$ i.e. $S(E) \subset N$. This case is illustrated in Figure 7; it is a rather general case which may occur in economies with many individuals and with many commodities. When all individuals have positive income at some price $p \in S(\mathsf{E})$, then limited arbitrage is always satisfied since by definition in this case $\forall h, \partial D(\rho_h, \Omega_i) \supset R^N_{++}$ for all $h = 1, \ldots, H$.

3.4. Limited arbitrage for subeconomies

When $X = R^N$, we say that the economy E satisfies *limited arbitrage for any subset of k traders*, when for any subset $K \subset \{1, \ldots, H\}$ of cardinality $k \leq H$

$$(LA) \bigcap_{h \in K} D(\rho_h, \Omega_h) \neq \emptyset. \tag{9}$$

When $X = R^N_+$ the definition is

$$(LA) \bigcap_{h \in K} \partial D(\rho_h, \Omega_h) \neq \emptyset. \tag{10}$$

Theorem 3 in the Appendix establishes that a market economy E has limited arbitrage if and only if it has limited arbitrage for any subset of $k = N + 1$ traders where N is the dimension of the commodity space.

3.5. Limited arbitrage as a transversality condition

The condition of limited arbitrage applies to any convex consumption set $X \subset R^N$ which is bounded below and has the property that $y \in X$ and $z \geq y \Rightarrow z \in X$, used for example in Chichilnisky and Heal [13], [17]. For simplicity, assume that either X is a translate of a positive orthant or else that the boundary of X, ∂X, is a manifold of dimension $N - 1$. For example, ∂X could be defined locally by a smooth function $f: R^N \to R$. Consider the gradient vector $D\partial X(y)$ of the function f which defines ∂X in a neighborhood of $y \in \partial X$, and let $H(D\partial X(y))$ be the line in R^N defined by the vector $D\partial X(y)$. We say that a vector v is *transersal* to $H(D\partial X(y))$, denoted

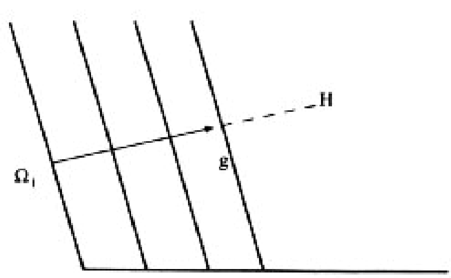

Figure 8. A more general consumption space in R^2. Limited arbitrage as a transversality condition.

$v \perp H(D\partial X(y))$ when the vector v is linearly independent from the subspace H; otherwise v is *not transversal* to H, denoted $v \dashv H$. We say that all supports $v \in S(\mathsf{E})$ are not transversal to an individual endowment Ω_h, denoted $v \dashv \Omega_h$, when $\Omega_h \in \partial X$ and $\forall v \in S(\mathsf{E})$, implies $v \dashv H(D\partial X(\Omega_h))$. The *limited arbitrage* condition is now defined as follows.

(LA) If $\forall v \in S(\mathsf{E})$, $v \dashv \Omega_h$ for some h, then there exists a price $p \in S(\mathsf{E})$ such that $\langle p, v \rangle > 0$ $\forall v \in A(\rho_h, \Omega_h)$, for all $h \in \{1, \ldots, H\}$.

The interpretation of this condition is that if every support in $S(\mathsf{E})$ fails to be transversal to some individual endowment, there is a supporting price at which only bounded increases in utility are affordable for all traders from initial endowments. Limited arbitrage can therefore be viewed as a transversality condition on the economy E.

4. Competitive equilibrium and limited arbitrage

This section establishes the main results linking the existence of a competitive equilibrium with the condition of limited arbitrage. A crucial part of the proof is to establish that with limited arbitrage the Pareto frontier of the economy is compact.

Theorem 1. *Consider the economy* $\mathsf{E} = \{X, \rho_h, \Omega_h, h = 1, \ldots, H\}$ *of Section 2, where* $H \geq 2$, $X = R^N$ *or* $X = R^N_+$ *and* $N \geq 1$. *Then the following two properties are equivalent:*

(*a*) *The economy* E *has limited arbitrage*

(*b*) *The economy* E *has a competitive equilibrium.*

Proof. We prove the results via four Lemmas. Lemma 1 proves that limited arbitrage is necessary for the existence of a competitive equilibrium. Next we establish that limited arbitrage is sufficient for the existence of a competitive equilibrium in three parts: Lemmas 2, 3 and 4 respectively. Lemma 2 establishes that the Pareto frontier of an economy with limited arbitrage is non-empty, closed and bounded away from the vector $(\sup_{\{x:x \in X\}} u_1(x), \ldots, \sup_{\{x:x \in X\}} u_H(x))$, (or simply a closed bounded set if we assume without loss of generality that $\forall h$, $\sup_{\{x:x \in X\}} u_h(x) = \infty$). This in turns implies, by the concavity preferences, that the Pareto frontier is homeomorphic to a simplex.[19] Lemma 3 is the proof of existence of a *pseudo or quasi equilibrium*; for this we use a fixed point argument on the Pareto frontier of the economy. Finally, using limited arbitrage we prove in Lemma 4 that the quasi equilibrium is also a competitive equilibrium.

Lemma 1. *Limited arbitrage is necessary for the existence of a competitive equilibrium in the economy* E.

Proof. Let the utility function $u_h: X \to R$ represent the preference $\rho_h \in \mathsf{E}$, i.e. for all x, $y \in X$, $u_h(x) > u_h(y) \Leftrightarrow x \succ_{\rho_h} y$. By appropriate renormalization and without loss of generality assume that $u_h(0) = 0$ so that $u_h(\Omega_h) \geq 0$, and that $\sup_{x \in X}(u_h(x)) = \infty$.

[19] A topological space X is *homeomorphic* to another Y when there exists a one-to-one onto map $f: X \to Y$ which is continuous and has a continuous inverse.

Now assume that (a) is not true, and consider the case $X = R^N(a)$ first. Then $\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset$, which implies that for all $y \in R^N$, there exists an $h \in \{1, \ldots, H\}$ and a vector $v(y) \in A(\rho_h, \Omega_h)$ such that $\forall \lambda > 0$

$$\langle y, \lambda v(y) \rangle \leq 0, \quad \text{and} \lim_{\lambda \to \infty} (u_h(\Omega_h + \lambda v(y)) = \infty. \tag{11}$$

Consider now a competitive equilibrium described by a price p^* and an allocation $(x_1^*, \ldots, x_H^*)$. By (11) for some $\lambda > 0$, $u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*)$ and $\langle p^*, \lambda v(y) \rangle \leq 0$, contradicting the fact that x^* is an equilibrium allocation. Therefore no competitive equilibrium exists when (11) is true: limited arbitrage is necessary for the existence of a competitive equilibrium when $X = R^N$. A similar proof of case (b) is in [16]. Consider next the case $X = R^N_+$. Assume first that $\forall q \in S(\mathsf{E}) \exists h \in \{1, \ldots, H\}$ s.t. $\langle q, \Omega_h \rangle = 0$. Then if limited arbitrage is not satisfied $\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) = \emptyset$, which implies that $\forall q \in R^N$, $\exists h$ and $v(q) \in A(\rho_h, \Omega_h)$

$$\langle q, \Omega_h \rangle = 0, \quad \text{and } \forall \lambda > 0, \langle q, \lambda v(q) \rangle \leq 0. \tag{12}$$

$$\text{Since } v(q) \in A(\rho_h, \Omega_h), \lim_{\lambda \to \infty} (u_h(\Omega_h + \lambda v(y)) = \infty.$$

Consider now a competitive equilibrium price p^* and the corresponding allocation $(x_1^*, \ldots, x_H^*)$. Then $p^* \in S(\mathsf{E})$, and (12) implies that $\exists h$ s.t. for some $\lambda > 0$, $u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*)$ and $\langle p^*, \lambda v(y) \rangle \leq 0$, contradicting the assumption that p^* and $(x_1^*, \ldots, x_H^*)$ define a competitive equilibrium.

It remains to consider the case where $\exists q \in S(\mathsf{E})$ such that $\forall h \in \{1, \ldots, H\}$, $\langle q, \Omega_h \rangle \neq 0$. But in this case by definition $\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) \neq \emptyset$ since $\forall h \in \{1, \ldots, H\}$ $\partial D(\rho_h, \Omega_h) \supset R^N_{++}$, so that limited arbitrage is always satisfied when an equilibrium exists. □

4.0.1. Limited arbitrage is sufficient for the existence of a competitive equilibrium

The proof goes as follows. We utilize the standard method of proving first the existence of a *quasi-equilibrium* as defined in Section 2, using a fixed point theorem on the Pareto frontier[20] $P(\mathsf{E})$. The quasi-equilibrium is subsequently shown to be a competitive equilibrium, thus completing the proof. This proof must address two practical difficulties, one when the consumption set $X = R^N$, and a different one when $X = R^N_+$. Both difficulties are resolved by the limited arbitrage condition. The problem is as follows: when $X = R^N$ the Pareto frontier $P(\mathsf{E})$ may be empty because the utility obtained by the traders from their initial endowments may not attain a maximum over feasible allocations when there are no bounds on short sales cf. Fig. 5. This failure leads to the non-existence of a competitive equilibrium in well known cases; this problem of existence appears also in economies with infinitely many commodities, but when commodity spaces are infinite dimensional it can appear even if the consumption set is the positive orthant, see the examples in Chichilnisky and Heal [13]. In practical terms, the problem is that the Pareto frontier may not be homeomorphic to a unit simplex, a property which is essential

[20] Introduced by Negishi [27].

in the proof of existence of a quasi-equilibrium. The role of the limited arbitrage condition in this case is to ensure that the Pareto frontier is bounded and closed; together with the quasi concavity of preferences this implies that the Pareto frontier is homeomorphic to a unit simplex so that standard existence arguments can be invoked. For case (b) sufficiency was also established in [16] and [31].

A more standard difficulty arises when the consumption set is $X = R^N_+$. Here the Pareto frontier is always non-empty closed and bounded and a quasi-equilibrium exists. However, in this case the quasi-equilibrium may fail to be a competitive equilibrium. This is the type of problem which the conditions of resource relatedness and of irreducibility are meant to circumvent. The problem arises only when some individual has zero income at the quasi-equilibrium allocation and is illustrated in Figure 7 above. In this case, minimizing costs may not imply maximizing utility so that a quasi-equilibrium may fail to be a competitive equilibrium. This second potential failure of existence is also ruled out by the condition of limited arbitrage.

Lemma 2. *The Pareto frontier $P(\mathsf{E})$ of the economy E is compact and homeomorphic to a unit simplex, when $X = R^N$ and when $X = R^N_+$.*

Proof. In the Appendix.

Example 5. *Figure 9 shows that our conditions cannot be weakened: they are necessary. It shows that the utility possibility set need not be bounded when the gradients of an indifference surface do not form a closed set. Figure 9 exhibits two indifference surfaces whose gradients asymptote to a gradient G which is never achieved at any allocation in either indifference surface. If there are two traders, and their endowments are $\Omega/2$ as illustrated, this leads to feasible utility values in $U(\mathsf{E})$ which approach utility values v_1 and v_2, but never reach these. The Pareto frontier is not closed in this case, because our limited arbitrage condition is not satisfied. The condition that the set of gradients be closed is not by itself sufficient for the Pareto frontier to be bounded and closed; limited arbitrage is also needed for this. Two different linear preferences in R^N give*

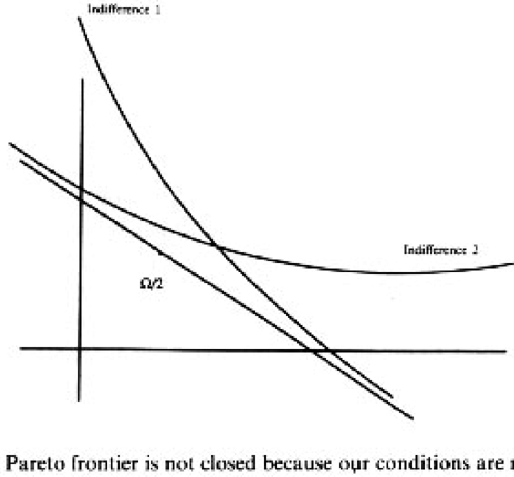

Figure 9. The Pareto frontier is not closed because our conditions are not satisfied.

rise to an unbounded frontier (all of R^N_+), yet the set of directions of gradients of any indifference surface is clearly closed, in fact a singleton.

Lemma 3. *Limited arbitrage implies the existence of a quasi-equilibrium in the economy* E *of Theorem* 1.

Proof: In view of Lemma 2, it is now standard to establish that a *quasi-equilibrium* always exists, either when $X = R^N$ or $X = R^N_+$ [27], [13]: for completeness we provide now a formal of existence of a quasi-equilibrium which works equally for these two cases next:

Define the set $T = \{y \in R^H : \sum_{h=1}^H y_h = 0\}$. For each $r \gg 0$ in Δ let $(x_1(r), \ldots, x_H(r)) \in F_\Omega$ now denote the feasible allocation which gives the greatest utility vector collinear with r:

$$(u_1(x_1(r)), \ldots, u_H(x_H(r))) = \sup_{w \in S_r} (u_1(w_1(r)), \ldots, u_H(w_H(r))),$$

in the vector order of R^H, and $\sum_{i=1}^H (x_i(r) - \Omega_i) = 0$. Such an allocation always exists because $\forall r \in \Delta$ S_r is bounded and closed by Lemma 2, it defines a non-zero utility vector which depends continuously on r. Now let

$$P = \{p \in R^N : \|p\| = 1\} \quad \text{and} \quad P(r) = \{p \in P : p \text{ supports } x(r)\}.$$

By standard arguments, $P(r)$ is not empty, see e.g. Chichilnisky and Heal [13], Lemma 2. Define now a map $\varphi : \Delta \to T$:

$$\varphi(r) = \{\langle p, \Omega_1 - x_1(r)\rangle, \ldots, \langle p, \Omega_H - x_H(r)\rangle : p \in P(r)\}$$

$\varphi(r)$ is a non-empty convex valued correspondence, $\sum_{h=1}^H z_h = 0$ if $z \in \varphi(r)$, and

$$0 \in \varphi(r) \Leftrightarrow (x^*, p^*) \text{ is a quasi-equilibrium, where } r = r(x^*) \text{ and } p^* \in P(r).$$

The next step is to show that φ is upper semi-continuous, i.e. if $r^n \to r$, $z^n \in \varphi(r^n)$, $z^n \to z$ then $z \in P(r)$. Consider the feasible allocation $x(r)$, where $r = \lim_n(r^n)$. Let v be any other allocation satisfying $u_h(v_h) > u_h(x_h(r))$, where $x_h(r)$ is the h-th coordinate of the vector $x(r)$ and v_h is the h-th coordinate of the vector v. Let $z^n \in \varphi(r^n)$ and $p^n \in P(r^n)$. Since $r^n \to r$, eventually $u_h(v_h) > u_h(x_h(r^n))$ so that $\langle p^n, v_h\rangle \geq \langle p^n, x_h(r^n)\rangle = \langle p^n, \Omega_h\rangle - z^n_h$, where z^n_h is the h-th coordinate of z^n: this follows from the definitions of z^n and p^n. Let $\{p^n\}$ be a sequence of vectors such that $p^n \in P(r^n)$. The set P is compact and $\bigcup_r P(r)$ is closed; therefore $\bigcup_r P(r)$ is compact as well. There exists therefore a vector $p \in P$ and a subsequence $\{p^m\}$ of $\{p^n\}$ such that $\langle p^m, v_h\rangle \to \langle p, v_h\rangle$, so that in the limit $\langle p, v_h\rangle \geq \langle p, \Omega_h\rangle - z_h$. Since this is true for all such v, it is also true for v satisfying $u_h(v_h) \geq u_h(r))$ and in particular for $v = x$ so that $\langle p, x_h\rangle \geq \langle p, \Omega_h\rangle - z_h$ implying that $z \in \varphi(r)$ as we wished to prove. The proof of existence of a quasi-equilibrium is completed by showing that φ has a zero cf. [27], [13]. For all $r \in \Delta$ define $\theta(r) = r + \varphi(r)$. The map $\theta : \Delta \subset \Delta$ is non-empty, upper semi continuous, convex-valued correspondence and it satisfies appropriate boundary conditions. By Kakutani's fixed point theorem, θ must have a fixed point r^* which is a zero of the map φ. The allocation $x^* = x^*(r^*)$ and a price $p^* \in P(r^*)$ define a quasi-equilibrium of the economy E. The proof of existence of a quasi-equilibrium just provided is equally valid when $X = R^N$ or when $X = R^N_+$. □

To complete the proof of the theorem it remains only to show that with limited arbitrage, the quasi-equilibrium is a competitive equilibrium.

Lemma 4. *Limited arbitrage implies the existence of a competitive equilibrium in the economy* E *of Theorem* 1.

Proof. In view of Lemma 3 it suffices to prove that a quasi equilibrium is a competitive equilibrium. Consider first the case $X = R^N$. Then $\forall h = 1, \ldots, H$ there exists an allocation in X of strictly lower value than x_i^* at the price p^*. Therefore by Lemma 3, Chapter 4, page 81 of Arrow and Hahn [2], the quasi equilibrium is also a competitive equilibrium. This establishes the existence of a competitive equilibrium when limited arbitrage is satisfied and $X = R^N$.

Now consider the case $X = R^N_+$. We have shown that when limited arbitrage is satisfied the economy E has a quasi-equilibrium consisting of a price p^* and an allocation x^*. It remains to show that the quasi-equilibrium is also a competitive equilibrium.

First note that if at the quasi-equilibrium (p^*, x^*) every individual has a positive income, i.e. $\forall h = 1, \ldots, H \ \langle p^*, \Omega_h \rangle > 0$, then by Lemma 3, Chapter 4 of Arrow and Hahn [2] the quasi-equilibrium is also a competitive equilibrium. Furthermore, since the quasi equilibrium $p^* \in S(\mathsf{E})$, then the set $S(\mathsf{E}) \neq \emptyset$. To prove existence we consider two cases: first, the case where $\exists q^* \in S(E)$: $\forall h, \langle q^*, \Omega_h \rangle > 0$. In this case, by the above remarks, (q^*, x^*) is a competitive equilibrium.

The second case is when $\forall q \in S(\mathsf{E}) \ \exists h \in \{1, \ldots, H\}$ s.t. $\langle q, \Omega_h \rangle = 0$, a case where the vectors q and Ω_h must have some zero coordinates. The limited arbitrage condition in this case implies

$$\exists q^* \in S(\mathsf{E}): \forall h, \ \langle q^*, v \rangle > 0 \text{ for all } v \in A(\rho_h, \Omega_h). \tag{13}$$

Let $x^* = x_1^*, \ldots, x_H^*$ be a feasible allocation in Υ supported by the vector q^* defined in (13). Then by definition, $\forall h, x_h^* \geq_{\rho_h} \Omega_h$ and q^* supports x^*.

Recall that any h minimizes costs at x_h^* because q^* is a support. Now, (q^*, x^*) can fail to be a competitive equilibrium only when for some h $\langle q^*, x_h^* \rangle = 0$, for otherwise the cost minimizing allocation is also utility maximizing in the budget set $B_h(q^*) = \{w \in X : \langle q^*, w \rangle = \langle q^*, \Omega_h \rangle\}$. It remains therefore to prove existence when $\langle q^*, x_h^* \rangle = 0$ for some h. Since by the definition of $S(\mathsf{E})$, x^* is individually rational, i.e. $u_h(x_h^*) \geq u_h(\Omega_h)$, it follows that when $\langle q^*, x_h^* \rangle = 0$, then $\langle q^*, \Omega_h \rangle = 0$, because q^* is a supporting price for x^*. If $\forall h$, $u_h(x_h^*) = 0$ then $x_h^* \in \partial R^N_+$, and by the monotonicity and quasi-concavity of u_h, any vector $y \in B_h(q^*)$ must also satisfy $u_h(y) = 0$, so that x_h^* maximizes utility in $B_h(q^*)$, which implies that (q^*, x^*) is a competitive equilibrium. Therefore (q^*, x^*) is a competitive equilibrium unless for some h, $u_h(x_h^*) \neq 0$.

Assume then that (q^*, x^*) is not a competitive equilibrium. Then for some h with $\langle q^*, \Omega_h \rangle = 0, u_h(x_h^*) \neq 0$, and therefore an indifference surface of a positive commodity bundle of u_h intersects ∂X at $x_h^* \in \partial X$. Let r be the ray in ∂X containing x_h^*. If $w \in r$ then $\langle q^*, w \rangle = 0$, because $\langle q^*, x_h^* \rangle = 0$. Since $u_h(x_h^*) > 0$, by Assumption 1 on u_h, all other indifference surfaces of u_h with higher utility intersect r, so that $y \in A(\rho_h, \Omega_h)$.

But this contradicts the choice of q^* as a supporting price satisfying (13) since

$$\exists h \text{ and } w \in A(\rho_h, \Omega_h) \text{ such that } \langle q^*, y \rangle = 0. \tag{14}$$

The contradiction between (14) and (13) arises from the assumption that (q^*, x^*) is not a competitive equilibrium. Therefore (q^*, x^*) must be a competitive equilibrium, and the proof of the theorem is complete. □

4.1. Subeconomies with competitive equilibria

Having completed the proof of the main result which establishes that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, it seems useful to point out that the condition of limited arbitrage need only be satisfied on subeconomies with no more traders than the number of commodities plus one. This is the next theorem:

Theorem 2. *Consider a market economy* E *as in Theorem* 1. *The following four properties are equivalent:*

(*a*) E *has a competitive equilibrium*

(*b*) *Every sub economy of* E *with at most* $N+1$ *traders has a competitive equilibrium*

(*c*) E *has limited arbitrage*

(*d*) E *has limited arbitrage for any subset of traders with no more that* $N+1$ *members.*

Proof: The proofs that (a)⇔(c) and that (b)⇔(d) follow directly from Theorem 1. That (c)⇔(d) follows from Theorem 3 in the Appendix. □

5. Social diversity and the existence of competitive equilibrium

It seems useful to situate the results of the previous section in the context of the literature on the existence of a competitive equilibrium, to discuss how the limited arbitrage assumption resolves the problem of non-existence, and how it is related to social diversity, cf. [16].

5.1. Related literature with bounds on short sales

As already pointed out, not all Arrow-Debreu exchange economies have a competitive equilibrium, even when all individual preferences are smooth, concave and increasing, and even when the consumption sets are positive orthants, see for example Arrow and Hahn [2], Chapter 4, p. 80. Their example is of interest because it is based on the diversity of endowments and preferences of the individuals in the economy: this diversity leads to a failure of continuity of the demand function. With a discontinuous demand, a competitive equilibrium generally fails to exist. "*This discontinuity will necessarily occur in some part of the price space, except in the unrealistic case in which the household has a positive initial endowment of all goods*" (Chapter 4, p. 80, [2]).

Of course, there are other lesser concepts of market equilibrium, which have the advantage that equilibrium allocations always exist when preferences are continuous and concave and the individuals' consumption sets are positive orthants, a property that the competitive equilibrium does not share. For example *quasi-equilibrium*, or *compensated equilibrium*, [2]. These are closely related concepts which define allocations where individuals minimize cost rather than maximizing utility. When prices and all individuals' incomes are strictly positive, these concepts agree with the competitive market equilibrium (Arrow and Hahn [2], Chapter 4). However, as Arrow and Hahn point out, the conditions that all prices are strictly positive, or that all individuals should have strictly positive endowments of all goods is unrealistic (2], Chapter 4, p. 80, para. 4), so that quasi-equilibrium or compensated equilibrium allocations will not be competitive equilibrium allocations in general. This technical issue has major welfare implications.

Economies with a competitive equilibrium stand alone in terms of their welfare properties: quasi-equilibrium (pseudo-equilibrium), or compensated equilibrium allocations are not generally Pareto efficient, as is the competitive equilibrium. Therefore the main justification for using market allocations, which is efficiency, would be lost unless we remain within the confines of a competitive equilibrium. For this reason we concentrate here on competitive equilibrium allocations.

The problems of non-existence of a competitive equilibrium are somewhat different when the consumption set X is the whole Euclidean space – i.e. when there are no bounds on short sales – than when the consumption set X is the positive orthant. When $X = R^N$ the limited arbitrage condition ensures that individuals, who must typically be diverse in order to achieve gains of trade, are not too diverse, so their desired trades can be accommodated within the same economy [16]. For example, there must exist a degree of consistency between traders' global cones at the initial endowments: a price hyper plane must exist leaving all the global cones to one side, the same for all traders, so that from initial endowments, no individual can afford allocations which lead to unbounded utility at these prices. When the consumption space is $X = R^N_+$, the failure of existence is somewhat different. Figure 7 illustrates a failure of the limited arbitrage condition in this case. What limited arbitrage does is to limit precisely the degree of diversity among the agents of the economy so that market equilibrium will exist.

Indeed for $X = R^N_+$, such limits on diversity are implicit in Arrow's resource relatedness [2] and in McKenzie's irreducibility condition [24], [25], [26]. All these conditions ensure that the endowments of any household are desired, directly or indirectly, by others, so that their incomes cannot fall to zero. Under both of these conditions, our limited arbitrage condition is always satisfied.

Irreducibility and *resource relatedness* conditions ensure that at a quasi-equilibrium or at a compensated equilibrium, all individuals' incomes are strictly positive, or strictly large than the minimum possible income. When individuals' incomes are all positive all the notions of equilibrium coincide. The problem of maximizing utility subject to a budget constraint, which is the competitive equilibrium condition, is then identical to that of minimizing the cost of an allocation with a certain utility level, which is the condition which defines a compensated equilibrium. Thus a quasi equilibrium, which always exists when preferences are concave and continuous and the commodity space is the positive orthant, Negishi [27], is also a competitive equilibrium.

The key to the conditions of Arrow, Debreu and McKenzie is to eliminate minimum income allocations. Yet traders with zero or minimum income do not by themselves rule out the existence of a competitive equilibrium see Figure 4A. An allocation where some individuals have zero, or the minimum possible, income, reflects a real situation: the fact that some individuals are considered worthless, they have nothing to offer that others want. Such a situation could be a competitive equilibrium. It seems realistic that markets could lead to such allocations: one observes them all the time in city ghettos. Our condition of *limited arbitrage* does not attempt to rule out individuals with minimum income; instead, it seeks to determine if society's evaluation of their worthlessness is shared. Individuals are diverse in the sense of not satisfying limited arbitrage, when someone has minimal income. This requires in turn that some individuals have minimal quantities of some goods – and, in addition, that there is no agreement about the value of those who have minimal income.

In sum: our condition of limited arbitrage is geometric in nature: it admits an interpretation as a transversality condition. It bounds the extent of diversity among the market's traders (Chichilnisky [15]), and their gains from trade (Proposition 2), but it does so in a different way than *irreducibility* [24], [25], [26] and *resource relatedness* [2]. The latter two are only applicable to economies where the consumption is bounded below – where there is a bound on short sales – and are not necessary for existence. Instead, limited arbitrage is applicable both to this case and also to the case where short sales are allowed, and is necessary and sufficient for existence of an equilibrium.

Finally we consider the condition that indifference surfaces of preferences of positive consumption bundles should be in the interior of the positive orthant, e.g. Cobb-Douglas: this implies that the set of directions along which the utilities increase without bound from initial endowments is the same for all traders. This condition implies that all individuals agree on choices with large utility values, again a form of similarity of preferences.

5.2. Related literature without bounds on short sales

Two conditions which have been used in economies with short sales $X = R^N$ are *no arbitrage* and *Condition C*. The former is used in finance; the connection between limited arbitrage and no-arbitrage was discussed in Section 3.2.

Condition C of Chichilnisky and Heal [13] is sufficient for the existence of a competitive equilibrium, but it is not necessary; it requires that, if along a sequence of allocations the utility of one of the traders increases beyond bound, then there exists another trader whose utility eventually decreases below the level of this trader's initial endowment along this sequence. Condition C applies to economies without bounds on short sales, where the consumption set is the whole Euclidean space; instead our limited arbitrage condition applies to economies with or without bounds on short sales. Formally, the cone used here to define limited arbitrage is strictly contained in general in the set of unbounded feasible allocations which appears in Condition C. This makes our condition of limited arbitrage strictly weaker than Condition C.

The condition of *no-arbitrage* used in [29] and [31] equals limited arbitrage when $X = R^N$ case (b), i.e. indifferences without half lines, but not generally; otherwise it is sufficient but not necessary for the existence of an Arrow-Debreu equilibrium. It is binding only when the consumption space is not bounded below: otherwise it is automatically satisfied. Condition [P] of [29] eliminates the formation of trading groups or coalitions in which members engage in unbounded and preference increasing trades (p. 398, [29]); condition [P] is generally different from limited arbitrage, see also Chichilnisky [16], because the global cones $A_h(\rho, \Omega)$ which define limited arbitrage are quite different in case (a) from the cones used in [29]. Global cones $A_h(\rho, \Omega)$ consist of rays which intersect every indifference surface of an individual's preference corresponding to utility values above that of the initial endowment. Instead, the "recession" cones introduced by Rockafeller and used by Page and by Werner to define no-arbitrage need not satisfy this condition [29], [31]; their recession cones are generally larger than our global cones, so that their duals are strictly smaller than our market cones, and therefore their non-empty intersection is a stronger assumption. This makes limited arbitrage a weaker condition, and one which is necessary as well as sufficient for the existence of a competitive equilibrium, while the conditions in [29] and [31] are not necessary for existence in general.

Limited arbitrage depends on endowments as well as preferences: with the same preferences and $X = R^N_+$ our economy will satisfy limited arbitrage for certain initial endowments of the traders and not for others. Indeed, one expects that the similarity of individuals should be defined in terms of their endowments as well as in terms of their preferences. The existence of a competitive equilibrium should also generally depend not only on individuals' preferences, but also on their endowments, and this is precisely what limited arbitrage shows when $X = R^N_+$. In contrast, Werner's cones are assumed to be the same at every allocation ([31], Assumption A3 and Proposition 1), so that no-arbitrage must be verified in principle at all allocations [31]. Finally, the conditions of Page and Werner are binding only when consumption sets are not bounded below and they are always satisfied otherwise ([31], Section 6, p. 1414) while, as already pointed out, limited arbitrage is binding whether consumption sets are bounded below or not.

6. Conclusions

We have shown that limited arbitrage is a necessary and sufficient condition for the existence of a competitive equilibrium in Arrow-Debreu economies with or without bounds on short sales. The same condition – limited arbitrage – was shown elsewhere [14] to limit voting cycles, to be necessary and sufficient for the existence of a continuous anonymous social choice map respecting unanimity on the space of all preferences which are similar to those of the traders in the economy **E**, and also to be necessary and sufficient for the existence of the core [16]. In this sense, the results of this paper and of [14], [15], [16] unify four forms of resource allocation: by markets, by financial arbitrage, by social choice, and by cooperative game theory (the core), which have developed separately and remain separate until now.

We have chosen competitive equilibrium allocations – and no other forms of equilibrium – because of the Pareto efficiency of competitive equilibrium, a property which is generally lost in weaker forms of market equilibrium, such as quasi-equilibrium and compensated equilibrium.

The interpretation of the limited arbitrage condition is somewhat different with or without short trades, although mathematically they are very similar. In the latter case it measures social agreement about allocating minimal value to the endowments of certain members of society, and this agreement must include those same individuals to which society assigns minimal value. It may seem surprising that such an agreement could exist. In the case that it does not, resource allocation breaks down: the competitive market has no competitive equilibrium there is no core and a social choice map does not exist.

The connection between the existence of a competitive equilibrium and the manipulation of market games would be a natural extension of these results. This could follow from the connection between the existence of social choice maps and the manipulation of games, Chichilnisky [11]. It also seems possible to extend the results of this paper to economies with production. Issues of survival and under-employment in market economies are also directions in which to extend the inquiry of this paper.

7. Appendix

7.0.1. Proof of Lemma 2: The Pareto frontier $P(\mathsf{E})$ is homeomorphic to a unit simplex when limited arbitrage is satisfied, $X = R^N$ or $X = R^N_+$

When the indifference surfaces of the preferences are bounded below, the proof is identical to that of the case where $X = R^N_+$, a case where the result is known (see Arrow and Hahn [2]). Therefore, we need only consider the case where $X = R^N$ and where the indifference surfaces of the traders are not all bounded below. The proof is by contradiction. Assume E has limited arbitrage. If $U(\mathsf{E})$ were *not* bounded there would exist a sequence of net trades $(z_1^j, \ldots, z_H^j)_{j=1,2\ldots}$ and $J \subset \{1, \ldots, H\}$ s.t. $\lim_j \sum_{k \in J} z_k^j = 0$, $\| z_k^j \| \to \infty \Leftrightarrow k \in J$, and $\lim_{j \to \infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for some $h = g$.[21] It suffices to consider the case where $\lim_{j \to \infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for all $h \in J$, which contradicts limited arbitrage.[22] To elaborate, consider two exhaustive and exclusive cases: Case 1 and Case 2.

Case 1: For infinitely many j's, $z_h^j \in A_h$ for all $h \in J$. Limited arbitrage requires that there exists hyperplane that leaves all the cones A_h one one side for all h, and this case would contradict the fact that $z_h^j \in A_h$ for all $h \in J$ and $\lim_j \sum_{h \in J} z_h^j = 0$, see ftn. 22. Since the contradiction arises from the assumption that $U(\mathsf{E})$ is unbounded, $U(\mathsf{E})$ must be bounded in this case.

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[21] Note that $\alpha_g = \lim_j \frac{z_g^j}{\| z_g^j \|} \in A(\rho_g, \Omega_g)$ denoted also A_g.

[22] $k \in J \Rightarrow \| z_k^j \| \to \infty$ so that $\forall k \in J$, $\alpha_k = \lim_j \frac{z_k^j}{\| z_k^j \|} \in \bar{A}_k$, for otherwise $\lim u_k(\Omega_k + z_k^j) < 0$; since $\lim_j \sum_{k \in J} z_k^j = 0$, $\alpha_g \in A_g$ (ftn. 21) and A_g is open, $\exists \{\gamma_k\}_{k \in J} : \sum_k \gamma_k = 0$ and $\gamma_k \in A_k \forall k \in J$.

Case 2: From some j onwards, $z_h^j \notin A_h$ for some $h \in J$. Consider the sequence $\{z_h^j \| z_h^j \|\}_{j=1,2,\ldots} \subset S^{N-1}$, the $N-1$ sphere in R^N. Since S^{N-1} is compact there exists a subsequence, denoted also $\{z_h^j / \| z_h^j \|\}_{j=1,2,\ldots}$ such that $\lim_{j\to\infty} z_h^j / \| z_h^j \| = \alpha_h \in S^{N-1}$ for all $h \in J$. Assume first that $\alpha_h \notin A_h$. Since $\alpha_h \notin A_h$, it follows that $Sup_{\lambda \in R^+}(u_h(\Omega_h + \lambda\alpha_h) < \infty$. This, together with the assumption on the utilities, implies that if Γ is the halfline defined by the vector α_h, either (a) $\exists w \in \Gamma$ where the gradient $Du_h(w)$ is orthogonal to Γ, or else (b) the utility u_h asymptotically approaches a maximum on Γ, or achieves a maximum at some $y \in \Gamma$, and is a constant on Γ beyond y. These two alternatives (a) and (b) are exhaustive when $\alpha_h \notin A_h$, and I will show that in both it is impossible that $\alpha_h = \lim_j z_h^j / \| z_h^j \|$ with $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) = \infty$. If the gradient $Du_h(w)$ is orthogonal to Γ at some point $w \in \Gamma$, and for $\lambda > 1$ $Du_h(\lambda w)$ projected on Γ is negative, then it is also negative in a neighborhood. Therefore for directions β_h sufficiently close to α_h, $\exists K > 0$ such that if V is the ray defined by the direction β_h, $\sup_{x\in V}(u_h(x)) < K$ on V, a contradiction. Therefore alternative (a) is not possible when $\alpha_h \notin A_h$. The second alternative (b) is that $\alpha_h \notin A_h$ and the utility u_h approaches a maximum or achieves a maximum over the halfline Γ at w, say $u_h(w) = n_h$, and remains constant thereafter on Γ. Recall that we are in Case 2, so that by assumption from some j onwards, $z_h^j \notin A_h$ for some h. It follows that for all directions β_h sufficiently close to α_h there is an h such that u_h approaches a maximum or achieves a maximum, say the value m_h, on the ray defined by β_h and remains a constant thereafter. This implies that the sets $\{u_h^{-1}(m_h)\}$ and $\{u_h^{-1}(n_h)\}$ asymptotically contain each a different halfline, namely the lines defined by the directions β_h and α_h respectively. Therefore the sets $\{u_h^{-1}(m_h)\}$ and $\{u_h^{-1}(n_h)\}$ contain elements which are at arbitrarily large distance from each other. But this is a contradiction with Assumption 2 of Section 2, which requires that for each h the preference u_h satisfies $\exists \varepsilon > 0 : \forall x \in X, \| Du_h(x) \| > \varepsilon$; which implies that the distance between two indifference surfaces is bounded, and in particular that the distance between $\{u_h^{-1}(m_h)\}$ and $\{u_h^{-1}(n_h)\}$ is bounded. Therefore alternative (b) leads also to a contradiction when $\alpha_h \notin A_h$. Since these two alternatives (a) and (b) are exhaustive, and each leads to a contradiction, it is not possible in Case 2 that from some j on $z_h^j \notin A_h$ for some h and simultaneously that $\alpha_h \notin A_h$. Therefore, in Case 2, there must exists an h such that $z_h^j \notin A_h$ for all j from one onwards and for all such h's, $\alpha_h \in A_h$. Now consider all those k for which $z_k^j \in A_k$; then by construction $\alpha_k \in \bar{A}_k =$ the closure of the set A_k. To summarize, the situation in Case 2 is as follows: for at least one h, $\alpha_h \in A_h$, and for all other $k, k \neq h, \alpha_k \in \bar{A}_k$. Since the cones A_h are open by Proposition 1, this implies that in Case 2 there exist vectors $\{\beta_h\}_{h\in J}$ sufficiently close to $\{\alpha_h\}_{h\in J}$ s.t. $\sum_h \beta_h = 0$ and $\beta_h \in A_h$ for all h, contradicting limited arbitrage. Limited arbitrage thus implies that $U(E)$ is bounded both in Case 1 and Case 2, and so is $P(E) \subset U(E)$.

We now establish that the Pareto frontier $P(\mathsf{E})$ is closed when $X = R^N$ and limited arbitrage is satisfied. The proof used here follows a similar line to that in Lemma 5, p. 375 of Chichilnisky and Heal [13]; their Lemma 5 also proves the closedness of the Pareto frontier in an economy without bounds on short sales. But the conditions on preferences used here are strictly weaker[23] than those used in [13], so the result established here is strictly stronger.

[23] Chichilnisky and Heal [13] require a condition (C) which is strictly stronger that our condition of limited arbitrage. This is discussed in Section 5.2 of this paper.

For any $r \in \Delta$, let $v = (v_1, \ldots, v_H) \in R^H_+$ be the supremum[24] of the set S_r, denoted also $v = \sup_{S_r}$; we know that such a v exists because the utility possibility set $U(\mathsf{E})$ is bounded.[25] Consider now a sequence of Pareto efficient allocations in $P(\mathsf{E})$ which converges to a vector v in $U(\mathsf{E})$ which is maximal[26] in $U(\mathsf{E})$. For the set $P(\mathsf{E})$ to be closed, we must prove that this vector v is the utility vector corresponding to a feasible allocation, i.e. that $v \in U(\mathsf{E})$. Formally, there exists a sequence $\{z^n\} \subset \Upsilon$, $z^n = (z^n_1, \ldots, z^n_H)$, such that: $U^n = (u_1(z^n_1), \ldots, u_H(z^n_H)) \in S_{r^n}$, $\lim_n \{r^n\} = r$, and $\lim_n U^n = v = \sup_{S_r}$. We need to prove that v is in $U(\mathsf{E})$; for otherwise $P(\mathsf{E})$ would not be a closed set.

Since $U(\mathsf{E})$ is bounded, and each utility u_h is monotonic, there exists a vector of utility values $(U^1, \ldots, U^H) = (u_1(y_1), \ldots, u_H(y_H)) \in R^H$, where $(y_1, \ldots, y_H)$ may or not be a feasible allocation, such that $\lim_{n \to \infty} U^n = (U^1, \ldots, U^H) = v$. By standard arguments, since $\lim_n U^n = v$ and $v = \sup_{S_r}$, the directions of all the gradients of the sequence of the utilities must draw close to each other:

$$\forall h = 1, \ldots, H, \lim_{n,m} \left(\frac{Du_h(z^n_h)}{\| Du_h(z^n_h) \|} \right) - \left(\frac{Du_h(z^m_h)}{\| Du_h(z^m_h) \|} \right) = 0. \tag{15}$$

Define now the sequence of directions of these gradients, $\{s^n_h\}_{n=1,2,\ldots}$ where $s^n_h = Du_h(z^n_h)/\| Du_h(z^n_h) \| \in S^{N-1} \subset R^N$. Since S^{N-1} is compact, $\forall h$ there exists a point of accumulation of $\{s^n_h\}_{n=1,2,\ldots}$, which by (15) must be a gradient common to all $h = 1, \ldots, H$, denoted $s \in R^N$. Since for all h, $u_h(z^n_h) \to v_h$, then $\forall \varepsilon > 0, \exists T$ and $\exists w^n_h \in R^N$ such that $u_h(w^n_h) = v_h$ and

$$\left\| \frac{Du_h(z^n_h)}{\| Du_h(z^n_h) \|} - \frac{Du_h(w^n_h)}{\| Du_h(w^m_h) \|} \right\| < \varepsilon \text{ for } n > T.$$

Therefore without loss of generality we may choose the sequence $\{z^n\} = \{z^n_1, \ldots, z^n_H\}$ so that $\forall n$ and $\forall h$, $z^n_h = u_h^{-1}(v_h)$. By construction the sequence $\{Du_1(z^n_1)/\| Du_1(z^n_1) \|, \ldots, Du_H(z^n_H)/\| Du_H(z^n_H) \| \}_{n=1,2,\ldots}$ converges to a common direction $(s, \ldots, s) \in R^{N \times H}$.

Now we utilize Assumption 2 case (a) made on preferences in R^N in Section 2: that the set of directions of indifference surfaces is closed. Apply this to the sequence

$$\{Du_1(z^n_1)/\| Du_1(z^n_1) \|, \ldots, Du_H(z^n_H)/\| Du_H(z^n_H) \| \}.$$

It implies that there exists a vector $z = (z_1, \ldots, z_H) \in R^{N \times H}$ such that $z_h \in u_h^{-1}(v_h)$ and $Du_h(z_h) = \lambda_h s$ for some $\lambda_h > 0$.

It is now standard to show that $\sum_{h=1}^H z_h = \sum_{h=1}^H \Omega_h$ i.e. that $(z_1, \ldots, z_H) \in \Upsilon$. This follows immediately if the sequence of allocations $\{z^n_1, \ldots, z^n_H\} \subset R^{N \times H}$ is bounded or has a bounded subsequence. If not, define the set Q consisting of allocations (which may or not be feasible) attaining the utility levels $v_1, \ldots, v_H$ and having

[24] v may or not be in $U(\mathsf{E})$. Note that a ray $r \in \Delta$ is a one dimensional half space in the positive orthant of R^N, in which a complete order is defined in a standard fashion. A bounded set $S_r \subset r$ has a unique supremum in this order, which we denote $\sup_{S_r}$.

[25] Note that because preferences are continuous, concave and increasing, when the vector $v = \sup_{S_r}$ is in $U(E)$, then by standard arguments the corresponding feasible allocation in Υ is Pareto efficient, see e.g. Arrow and Hahn [2] p. 111.

[26] The vector v is said to *maximal* in $U(E)$ if there exists no $w \in U(E)$ such that $w \geq v$ with $w_h > v_h$ for some h. Note that v may be maximal in $U(E)$ even though $v \notin U(E)$.

gradients equal to s. Formally:

$$Q = \{(y_1, \ldots, y_H) \in R^{N \times H} : Du_h(y_h) = \lambda_h s \text{ and } v_h = u_h(y_h)\}.$$

Note that the set Q could be a singleton. Our assumption on preferences[27] implies that Q is not empty, since the vector $z = (z_1, \ldots, z_H) \in Q$. Furthermore, by the concavity of preferences, Q is a closed, convex subset of an affine subset of R^N. But the sequence of allocations $\{z_1^n, \ldots, z_H^n\}$ approaches the set Q as closely as desired, and is in Υ. This implies that Q and Υ are at zero distance.[28] Since both Q and Υ are closed convex subsets of affine spaces in $R^{N \times H}$ and they are at zero distance this implies that $\Upsilon \cap Q \neq \emptyset$. Therefore there exists a feasible allocation $(y_1, \ldots, y_H) \in \Upsilon \subset R^{N \times H}$ such $Du_h(y_h) = \lambda_h s$, and $v_h = u_h(y_h)$, as we wished to prove. We have therefore completed the proof that the Pareto frontier $P(\mathsf{E})$ of the economy E is closed in case (a). That $P(\mathsf{E})$ is not empty follows from the same proof, but starting from an arbitrary sequence $\{z^n\} \subset \Upsilon$ of feasible allocations (which are not necessarily Pareto efficient allocations) having utility values which converge to $v = \sup_{S_r}$. Case (b) is immediate [16].[29]

We have shown that limited arbitrage implies that, when $X = R^N$, the set $P(\mathsf{E})$ is closed and bounded. The proof that $P(\mathsf{E})$ is homeomorphic to the unit simplex $\Delta \in R^H$ is now standard from the quasi concavity of the preferences, see for example Arrow and Hahn [2], p. 111: their proof requires only concavity of preferences and the fact that the Pareto frontier $P(\mathsf{E})$ is closed and bounded. For the case $X = R^N_+$, their proof establishes directly that this Pareto frontier is always homeomorphic to the unit simplex. □

Theorem 3 is used in Theorem 2 in proving that the economy E has a competitive equilibrium if and only if limited arbitrage is satisfied on subsets of at most $N + 1$ traders, where N is the number of commodities in the economy. This condition simplifies the requirements of verifying limited arbitrage, restricting this to subsets of at most $N + 1$ trades in the economy, where N is the number of commodities.

Theorem 3. *Consider a family* $\mathbf{U} = \{U_i\}_{i=1,\ldots,H}$ *of convex sets in* $R^N, H, N \geq 1$. *Then*

$$\bigcap_{i=1}^{H} U_i \neq \emptyset \text{ if and only if } \bigcap_{j \in J} U_j \neq \emptyset$$

for any subset of indices $J \subset \{1, \ldots, H\}$ *having at most* $N + 1$ *elements.*

In particular, an economy E as defined in Section 2 satisfies limited arbitrage, if and only if it satisfies limited arbitrage for any subset of $k = N + 1$ traders, where N is the number of commodities in E.

Proof. See Chichilnisky [10]. □

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27 Assumption 2(a) of Section 2.

28 The distance between two sets in euclidean space is the infimum of the euclidean distance of any two of their points.

29 When $v = (v_1, \ldots, v_H) = \lim_j v^j, v^j \in P(E) \forall j, v \notin P(E)$, then $\exists h_1 \neq h_2 : (v_{h_i} - \Omega_{h_i}) \in \partial A_{h_i}, i = 1, 2$, contradicting limited arbitrage.

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