Topological Social Choice

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Topological aggregation of preferences: the case of a continuum of agents

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Abstract. This paper studies the topological approach to social choice theory initiated by G. Chichilnisky (1980), extending it to the case of a continuum of agents. The social choice rules are continuous anonymous maps defined on preference spaces which respect unanimity. We establish that a social choice rule exists for a *continuum* of agents if and only if the space of preferences is contractible. We provide also a topological characterization of such rules as generalized means or mathematical expectations of individual preferences.

1. Introduction

A classical problem of social organization is how to aggregate individual into social preferences. Typical examples are voting procedures; an acceptable procedure must satisfy certain properties, or "axioms". Arrow (1951) required that the map from individual to social preferences, the "social choice rule", should be: (i) non-dictatorial, so that the outcome is not decided solely by one individual, (ii) independent from irrelevant alternatives, and (iii) Pareto, namely it should respect the unanimous wishes of the individuals between any two choices. Even though these axioms seem reasonable, without restrictions on individual preferences no such social choice rule exists. This is Arrow's impossibility theorem.

It seems natural to consider other formulations of social choice which could lead to the existence of social choice rules. Among them, the topological approach introduced in Chichilnisky (1980, 1982) deals with a topological space of preferences and requires three axioms: the aggregation rules from spaces of individual to social preferences must be (i) continuous, (ii) anonymous, i.e. invariant under permutation of the individuals, and (iii) respect unanimity, so that if everybody shares the same preference, so does society. Chichilnisky (1980, 1982) showed that these axioms also lead to an impossibility theorem when preferences are unrestricted. However, by considering restricted spaces of preferences Chichilnisky and Heal (1983) established the following result: "A *preference space' admits a social choice rule for every finite number of individuals n >_ 2, if and only if it is contractible"*. This result, known as the *resolution of the social choice paradox*, refers to problems with finitely many individuals. It reveals the intrinsic topological structure of the problem of social choice, because it shows that social choice rules exist if and only if the preference space is contractible, i.e. if and only if their topology is trivial.

Chichilnisky and Heal (1979) studied also a natural extension of the problem to infinitely many individuals; they constructed social choice rules for problems with countably many agents, rules which are Pareto and non dictatorial. These were constructed by taking the social preference to be a limit of sequences of individual preferences, or appropriate extensions when such limits do not exist. A *sequential passage to the limit* of the aggregation rules for finite individuals was also considered in Candeal et al. *(1992)*. More recently, Chichilnisky (1996) introduced a concept of non-dictatorship of the present and non-dictatorship of the future in the context of economies with infinitely many generations, and established the existence and a full characterization of continuous social choice rules satisfying continuity and these two axioms.

In this paper we consider the problem of topological aggregation of preferences for a *continuum* of individuals. A continuum of individuals reflect the idea of a very large number of participants, among which each individual has a. negligible impact, an idea which is central to the theory of competitive markets. A pioneer paper in this framework is Aumann (1964), where markets with a continuum of traders were studied. The problem of social choice involving a continuum of agents has not, however, been considered in the literature before.

Using a continuum framework, we establish the main result of this paper. We establish that there exists a social choice rule for a continuum of individuals if and only if the space of preferences is contractible. This is similar to the above result by Chichilniksy and Heal's (1983) on the topological structure of the social choice problem. Furthermore we establish that the social choice rules we construct are an extension of generalized averages: indeed, they are obtained by integrating or obtaining a mathematical expectation of individual's preferences.

2. Basic concepts and previous results

Basic concepts of social choice are the set of individual preferences, denoted X, and a set of individuas [2] or voters. The space of preferences X is a Hausdorff [3] topological space. The space of profiles of preferences, namely of listings of all individual preferences, is X" when there are *n* individuals, and $\prod_{i=1}^{\infty} X$

[1] Minimal regularity conditions are required on the space of preferences; for example it must be a parafinite CW-complex. This is a mild condition which includes all locally convex spaces such as manifolds, polyhedra, all finite simplicial complexes, and many infinite dimensional spaces such as spheres in Banach spaces

when there are countably many individuals. A social choice rule is a map from spaces of individual to social preferences, $\phi: X^n \to X$ with n individuals, and $\phi: \prod_{i=1}^{\infty} X \to X$ with countably many.

We shall assume that social choice rules respect two normative properties, anonymity and respect of unanimity. Also an assumption of continuity is required.[4]

A *n-Chichilnisky rule* (see Chichilnisky 1980, 1982) on X is a map from the product space X^n into X:

$$\phi: X^n \to X$$

$$(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n)$$

satisfying:

(i) CONTINUITY: ϕ is a continuous map (X^n is endowed with the product topology).
(ii) ANONYMITY: $\phi(x_1, \ldots, x_n) = \phi(x_{i_1}, \ldots, x_{i_n})$ for any rearrangement of {1, ..., n}.
(iii) RESPECT OF UNANIMITY: $\phi(x, \ldots, x) = x$ for every $x \in X$.

Similarly, in the case of an infinite but countable number of agents, one defines a *countable ∞-Chichilnisky rule on X* as a map:

$$X^\infty = \prod_{i=1}^{\infty} X \to X,$$

continuous, anonymous and unanimous. The concepts of anonymity and unanimity are defined similarly to the finite n-dimensional case.

The choice of topology can be important for the existence of continuous anonymous maps; anonymity is a symmetry property and as such it poses topological restrictions. Indeed, Efimov and Koshevoy (1994), and Lauwers (1997) have proved that the *product topology (or Tychonoff topology) on the space of profiles* leads to the non-existence of continuous anonymous rules respecting unanimity for countably many individuals. However if a *uniform topology* is considered on $\prod_{i=1}^{\infty} X$, then appropriate social choice rules exist. To avoid such problems one considers a weaker concept than anonymity:

(i) A *finitely anonymous countable ∞-Chichilnisky rule (or weak countable ∞-Chichilniksy rule)* on X is a map $\phi: X^\infty \to X$, with ϕ continuous, unanimous and *finitely anonymous.* That is, ϕ takes the same value on any two sequences such that the second is a finite rearrangement (only finitely many terms change their positions) of the former one.

As already mentioned, Chichilnisky and Heal (1983) established that when the set of individuals is finite, the existence of aggregation rules for all $n \geq 2$ is equivalent to extreme topological simplicity, indeed topological triviality. Thus, when the space of preferences X is a parafinite cellular complex *(parafinite CW-complex),* the existence of a n-rule for every n has as a necessary and sufficient condition the contractibility of X. This result, known as the

[4] For a motivation of the role of continuity in Social Choice theory see Chichilnisky (1982)

resolution of the Social Choice paradox is crucial in the framework of topological aggregation of preferences. The first proof of this fact appeared in Chichilnisky and Heal (1983) (see also Chichilnisky (1991) or Candeal and Induráin 1991).

With respect to the *countable infinite* case, there are several partial results (see Chichilnisky and Heal 1979 or Candeal et al. 1992). For example, we have:

(i) The existence of a countable ∞-Chichilnisky rule on a preference space X implies the existence, for every $n \in \mathbb{N}$, of a *n-Chichilnisky* rule on X. *Thus*, when X is a parafinite CW-complex, the existence, on X, of a countable ∞-Chichilnisky rule implies the contractibility of X.

(ii) Let X be a compact space of preferences. The existence on X of a *n-Chichilnisky* rule for every $n \in \mathbb{N}$ implies *the* existence of a weak countable ∞-Chichilnisky rule on X. Consequently, if X is a parafinite CW-complex, there exists on X a weak countable ∞-Chichilnisky rule.

(iii) Let X be a compact space of preferences. Then there exists a continuous, Pareto, non-dictatorial map, for countably many individuals, Chichilnisky and Heal (1979, 1994).

3. Topological Chichilnisky rules for a continuum of agents

We seek extensions of the above results to the case in which there is a continuum of individuals. To extend the notion of topological Chichilnisky rule for the case of a *continuum* of agents, we need to adequately express a *continuum Cartesian product* of copies of the preference space X. Obviously a continuum Cartesian product of X can be represented as X^I, the product of X with itself as many times as elements in I, where I has a suitable continuum cardinality, for example I could be the unit interval. The standard definition for this product space X^I is as the space of all functions from I to X.

The next step is the selection of a natural, suitable topology for X^I. We are interested in models leading to a result parallel to the *"Chichilnisky and Heal resolution of the Social Choice paradox"* [5]. In any case, the assumptions of unanimity, anonymity and continuity must be properly defined for the continuum case.

(i) Respect of unanimity.

The unanimity property means that if all the agents have the same preference (over all choices) then the social choice rule assigns to that profile that common preference. In other words, the restriction of the rule to the diagonal of the product space X^n [6], ϕ/Δ, is exactly the identity map, i.e. $\phi/\Delta = id$. To translate this property to the context of a continuum of agents, it suffices to require that the image under the social rule of *any constant map* with values in X (which is the equivalent of the diagonal 0 in the finite product

[5] A priori, we know nothing about the existence of continuum rules on suitable models. Of course, we do not know, at this stage, if such existence is related to additional topological properties, e.g. contractibility, on the preference space

$\Delta = \{(x_1, \ldots, x_n) \in X^n;\ \forall i, j,\ x_i = x_j\}$

space Xn) is just that constant value of the map. For this property to make sense, constant X-valued maps must be included in the functional space we consider as a model for the space of profiles.

(ii) Anonymity

With respect to anonymity, we require a sensible requirement: Suppose we have a partition of the space of individuals [0,1] into a finite number of coalitions, all having the same size, and that all the members of each coalition have the same partition. Then the rule must assigns the same value to any rearrangement of those over the same partition. We emphasize the condition on the partition that all its coalitions have the same size. Since the concept of size is needed to define this form of anonymity, a functional space involving some type of *measure will* be required. By technical reasons derived from the evaluation of some integrals we will work with the closed unit interval I = [0,1], or any other normalized finite measurable space equipped with the *Lebesgue measure*. Thus, by definition of anonymity, the simple functions' supported over sets with the same measure have to be included in our model for the set of profiles as well.

(iii) Continuity

As already mentioned, the key matter here is the continuity assumption. In view of the results by Efimov and Koshevoy (1992) and Lauwers (1997), one must consider a suitable topology. This is because the countable case is "included" in the continuum one, in the sense that a continuum of individuals certainly contains a countable number of individuals. '1111c: Qfore, a rule for a continuum of individuals induces a rule for countably many. As shown by Efimov and Koshevoy, the problem has no solution with the pointwise convergence topology. This implies that on the functional space X', one requires a topology finer than the pointwise-convergence topology, which is equivalent to the product topology in this context.

An adequate topology is the *compact-open topology*. This is the topology which has as subbasis the set of all functions which carry a given compact set K c I into a given open set U c X. This topology is identical with the topology of uniform convergence on compacta when X is a uniform space.'

Having in mind the previous discussion we formalize the problem as follows.

- The space of preferences X is a uniform Hausdorff topological space, for example a metric space, and the continuum of individuals is described by the unit interval [0,1] endowed with the usual topology and the Lebesgue measure μ.
- $\mathscr{S}([0,1],X)$ denotes the set of all simple maps, i.e., maps whose range is finite. A simple map s will be represented as follows:

$$s = x_1\chi_{A_1}, \ldots, x_n\chi_{A_n}$$

'Simple functions are maps whose range is a finite number of values

[8] For the definition of the compact open topology see Dugundji (1966), p. 257, and Kelley (1955, p. 221, Kelley (1955), p. 229, Theorem 1, proves that the compact open topology is the uniform topology on compacta when X is a uniform space, for example, when X is a metric space

where $(x_1, \ldots, x_n) \in X^n$, $(A_i)_{i=1,\ldots,n}$ is a partition of [0,1] into disjoint Lebesgue-measurable sets A_i, for i = 1, . . . , *n,* and $s(x) = x_i \Leftrightarrow x \in A_i$.

In particular, the range of s is the subset $\{x_1, \ldots, x_n\} \subset X$.

- The space of profiles, or list of individual preferences is the minimal closed space which contains the simple functions under the uniform convergence, i.e. it is the space of functions $f: [0,1] \to X$ for which there exists a sequence $(f_n)_{n \in N}$ of simple maps uniformly converging of f p-almost everywhere.[9] The space of preferences is denoted $\mathscr{L}_\infty([0,1], X)$. We endow $\mathscr{L}_\infty([0,1], X)$ with the relative compact-open topology μ-a.e.

Recall that the sets of the form

$S(K, U) = \{ f : [0,1] \to X; f(t) \in U, t \in C$, where

K is any compact subset of [0, 1], and U any open subset of X},

constitute a subbase for this topology.

Now we define the concept of a continuum Chichilnisky rule:

A *continuum Chichilnisky rule* on X is a map

$$\phi : \mathscr{L}_\infty([0,1], X) \to X$$

satisfying the following conditions:

(i) CONTINUITY: ϕ is continuous.

(ii) ANONYMITY: For any finite partition $(E_i)_{i=1,\ldots,n}$ of [0,1] with $\mu(E_i) = 1/n$ $(i = 1, \ldots, n)$, any n-tuple $(x_1, \ldots, x_n) \in X^n$, and any perturbation of the set $\{1, \ldots, n\}$, denoted $\sigma \in S(n)$:

$$\phi(x_1\chi_{E_1}, \ldots, x_n\chi_{E_n}) = \phi(x_{\sigma(1)}\chi_{E_1}, \ldots, x_{\sigma(n)}\chi_{E_n}),$$

(iii) Respect of UNANIMITY: For every $x \in X$:

$$\phi(x\chi_{[0,1]}) = x.$$

A first result states the relationship between continuum and finite Chichilnisky rules:

Proposition 1. The existence of a continuum *Chichilnisky rule* $\phi : \mathscr{L}_\infty([0,1], X) \to X$ *implies the existence, for every* $n \in N$,[10] *of a (finite) n-Chichilnisky rule on the space of preferences* X, $\phi_n : X^n \to X$.

Proof. Let $\phi : \mathscr{L}_\infty([0,1], X) \to X$ be a continuum Chichilnisky rule.

Given $n \in N$, let us define the map

$$\phi_n : X^n \to X$$

by

$$\phi_n(x_1, \ldots, x_n) = \phi(x_1\chi_{[0,1/n]}, x_2\chi_{[1/n,2/n)}, \ldots, x_n\chi_{[(n-1)/n,1]}) \in X.$$

[9] Except on a set of measure zero, denoted also μ – a.e.
[10] N is the set of integers

Clearly, ϕ_n is **well defined and it is anonymous and unanimous.** To **see that** ψ_n **is continuous, it is enough to verify the continuity of the** map O‡ : X" -> ~~ ([0,1], X) defined as

$$(x_1, \ldots, x_n) \qquad s_n = (\mathrm{x1X[O,\ 1/n]},\ x_2\chi_{[1/n,\,2/n)}, \ldots, x_n\chi_{[(n-1)/n,\,1]}) \in \mathscr{L}_\infty([0,1], X),$$

because ψ_n **is the composition of** ϕ_n^* with ϕ, i.e. ϕ_n = 0 - 0*, and ϕ is **continuous.** Denote by I_i = [(i - *1)/n, i/n), 1 < i < n.* **Notice that the preimage** $(\phi_n^*)^{-1}(S(C,U))$ of a subbasic subset equals *V*1 x ... x V_n, **with V; = U if C meets I**1 **p-a.e. and V**i **= X otherwise, and these sets are open in the product topology of X". Therefore** ϕ_n^*, **and hence** ϕ_n, **is continuous.** □

The following results are obtained for a wide class of preference spaces X called CW-complexes; these include euclidean spaces, all manifolds and polyhedra, and are constructed by pasting up properly a number of simple spaces called "cells". CW complexes contain a finite number of cells; examples are spheres, tori, balls, and cubes. Parafinite CW complexes may be infinite dimensional, as they may contain an infinite number of cells, although they may contain only a finite number of cells in each dimension.

Corollary *2. Let the space of preferences X be a parafinite CW-complex. If there is a continuum Chichilnisky rule on X, then X is contractible.*

Proof. **This is a direct consequence of Proposition 1 and the** *Chichilnisky and Heal resolution of the social choice paradox* **(see Chichilnisky and Heal 1983; Chichilnisky 1991, or Candeal and Indurain 1991).** □

The next result extends Corollary 2 to show that the contractibility of X is not only necessary but is also sufficient for the existence of social choice rules.

Theorem *3. Let the space of preferences X be a CW-complex. Then X admits a continuum Chichilnisky rule* $\phi : \mathscr{L}_\infty([0,1], X)$ *-+ X if and only if X is a contractible space.*

Proof. **Let X be a contractible CW-complex.** Let *K(X)* **be the closed convex hull of X. Then (X, K(X))** is a cellular pair." *K (X)* **is obviously contractible, since it is convex. Thus the inclusion mapping i from X into K(X) is a weak homotopy equivalence." Hence, it is an homotopy equivalence." Therefore X is a deformation retract of K(X).** [14] **By Whitney's theorem," we may consider** *K(X)* a **topological subspace of some Euclidean space R', for** *k* **big enough so that in particular X is a uniform space; let "+" denote the standard addition within R'`. Define now the map**

$$0 : \mathscr{L}_\infty([0,1], \mathrm{K(X)}) \to K(X)$$

$$f \to \theta(f) = \int_{[0,\,1]} f \,\mathrm{d}\mu.$$

[11] Rohlin and Fuchs (1981), p. 118
[12] Rohlin and Fuchs (1981), p. 445
[13] Rohlin and Fuchs (1981), p. 446
[14] Rohlin and Fuchs (1981), p. 119
is See, for instance, Brocker and Janich (1973) Satz 7.7

We first check that this map is *well defined.* Notice that any simple map $s = x_1\chi_{A_1}, \ldots, x_n\chi_{A_n}$ is, obviously, integrable; indeed its integral is $x_1\mu(A_1) + \cdots + x_n\mu(A_n) \in K(X)$.

Now if *f* is a uniform limit of simple functions, it is μ – *a.e.* bounded, and therefore it is a Lebesgue-integrable as a map from [0,1] to R'; in fact by the Lebesgue bounded convergence theorem, its integral is a limit in R' of elements that belong to K(X). But *K(X)* is closed in R'. Therefore *O(f)* must lie in K(X).

' Now since (X, K(X)) *is* a cellular pair, there exists a continuous *retraction"* from K(X) into X, denoted

$$\rho: K(X) \to X.$$

Consider therefore the following composition of maps 0 = j - 0 - p, which maps -°,,, ([0,1], X) into X:

$$\mathscr{L}_\infty([0,1],X) \xrightarrow{j} \mathscr{L}_\infty([0,1],K(X)) \xrightarrow{\theta} K(X) \to X$$

where j is the inclusion map. *This composition is a continuum Chichilnisky rule defined on X.* □

On parafinite contractible spaces, Chichilnisky topological n-rules are, essentially, retractions of convex means (see, e.g., Chichilnisky and Heal 1983). Here, working with the *continuum* case, we have obtained a parallel result:

Corollary 4. *Any continuous, anonymous social choice rule* $\phi: \mathscr{L}_\infty([0,1],X) \to X$ *respecting unanimity is homotopic to a mathematical expectation of individual preferences.*

Proof. By the previous theorem a map ϕ with the desired properties exists if and only if X is contractible. Note that the map $\phi = j \circ 0$ - p constructed above is a mathematical expectation i.e. it has the desired property. Furthermore, when X is contractible, any other map $\phi: \mathscr{L}_\infty([0,1], X) \to X$ is homotopic to j - 0 - p. This completes the proof.

Theorem 3 has been stated for *finite* CW-complexes (a finite union of cells of finite dimension). However, Chichilnisky and Heal's resolution of the Social Choice paradox was stated for the more general case of closed *parafinite CW-complexes,* consisting of an appropriate union of cells which could include infinitely many cells, but such that the number of cells of each finite dimension is finite. It is possible to strengthen Theorem 3 to show that the analogous result is valid for the parafinite case.

Theorem 5. *Let the space of preferences be a parafinite CW complex. Then X admits a continuum Chichilnisky rule* $\phi: \mathscr{L}_\infty([0,1],X)$ *if and only if X is a contractible space.*

Proof. The strategy of this proof is as follows. First we define a social choice map on simple profiles in Y,,, ([0,1], X) --> X, i.e. profiles defined by simple maps, and then show that this rule can be extended to all profiles $f \in \mathscr{L}_\infty([0,1],X)$,

[16] A retraction from a space Y into a subspace X c Y is a continuous map $r: Y \to X: \forall x \in X, r(x) = x$

since each f can be approximated uniformly by simple maps. The definition of the social choice rule on simple map is easy to achieve: by definition, the image of a simple map is contained in the union of all cells of X up to some finite dimension, say n, this union is called the n-skeleton of X, and is denoted X_n. The integral of each simple map is therefore contained in the convex hull of X,,, denoted K(Xn). The integral is shown to be well defined and it is a continuous, Pareto and non-dictatorial map on simple profiles. Next we show that the limit of these integrals defines the integral off; the map assigning to f its integral, which is the natural extension of the rule defined in Theorem 3, is the desired social choice rule and completes the proof.

Let s be a simple map in Y.([0,1], X). The image of s consists of finitely many points and is contained in some skeleton of X, say the *n-th* skeleton of X, Xn, so that its integral $\int_{[0,1]}$ s dp is contained in the convex hull of X,,, $K(X_n)$. Compose this integral with the retraction map ρ_n from K(Xn) into Xn defined in (1) above. This defines the value for the integral of s in X, denoted $\phi(s_n) \in X$

$$\phi_n(s) = \rho_n \circ \int_{[0,1]} s \, dp.$$

It is immediate to check that the above construction can be made independent from the choice of n-skeletons, by choosing inductively the retraction maps ρ_n from K(X„) onto Xn so that they agree on subskeletons, i.e. so that

$$X_n \subset X_q \Rightarrow \rho_n : K(X_n) \to X_n = \rho_{q/X_n} : K(X_n) \to X_n,$$

where ρ_{q/X_n} is the restriction of the map ρ_q to the subset $X_n \subset X_q$. One therefore has defined inductively a 'map p : $\cup_n K(X_n) \to X$, such that $\rho/K(X_n) = \rho_n$.

The composition map $\phi(s)$ = p - 1(o, 1] s dy defines a social choice map on the subspace of simple maps of $\mathscr{L}_\infty([0,1], X)$. The map ϕ is similar to the social choice map defined in Theorem 3; in particular it is continuous, non-dictatorial and Pareto. The next step is to extend ϕ to all maps in Y,,,, ([0,1], X).

Recall that any f in P,,,, ([0,1], X) is the uniform limit of simple maps $(s_n)_{n \in N}$ The integrals $\int_{[0,1]} s_n \, d\mu$ in fact define a Cauchy sequence in R^∞, i.e. given a convex neighborhood U of the origin in the space of all sequences with the product topology, R^∞, there exists $n_0 \in N$ such that

$$\int_{[0,1]} (s_n - s_m) \, d\mu \in U$$

for every ' $n, m > n_0$. Now $\int_{[0,1]} s_n \, dp \in K(X)$ for every n and K(X) is closed, so that

$$\int_{[0,1]} f \, d\mu = \lim_{n\to\infty} \int_{[0,1]} s_n \, dp \in K(X).$$

It is routine to check that this definition is consistent, i.e. it does not depend on the sequence of simple maps (s_n) chosen. Thus the map $\phi : \mathscr{L}_\infty([0,1], X) \to X$ given by $\phi(f) = \rho \int_{[0,1]} f \, d\mu$ is the desired social choice rule. □

Remark 1. The set Y,,,, ([0,1],K(X)) used in the previous discussion may be substituted by a larger set, that we shall denote Y1([0,1], K(X)) in the same way that $\mathscr{L}_\infty([0,1])$ is contained in $\mathscr{L}_1([0,1])$ and following the steps of the construction of the Lebesgue integral on [0,1].

Thus, we say that a map $f: [0,1] \to K(X)$ is *integrable,* if embedding $K(X)$ in $\mathbf{R}^k$, via Whitney's theorem, the map $f^*: [0,1] \to R^k$ is Lebesgue integrable, f^* equal to the composition $i \circ f$, with i the embedding map $i: K(X) \to R^k$ given by Whitney's theorem.

If f is integrable, its integral, denoted by $\int_{[0,1]} f$ will be the following element of K(X):

$$\int_{[0,1]} f = i^{-1} \int_{[0,1]} (i \circ f)\, d\mu$$

Now define the set $\mathscr{L}_1([0,1], K(X))$ as follows: $\mathscr{L}_1([0,1], K(X)) = \{ f: [0,1] \to K(X); f$ is integrable$\}$, and observe that Theorem 3 and its Corollary can be generalized using $\mathscr{L}_1([0,1], K(X))$ instead of $\mathscr{L}_\infty([0,1], K(X))$. □

Remark 2. Recently Lauwers and van Liedekerke (1993) have proved an analogue of the Chichilnisky-Heal resolution of the social choice paradox in the context of countably many agents. Their result is phrased in terms of a new concept of anonymity, that they call *bounded anonymous infinite rules.*

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