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Social Diversity, Arbitrage, and Gains from Trade: A Unified Perspective on Resource Allocation

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People trade because they are different. Gains from trade and the scope for mutually advantageous reallocation depend naturally on the diversity of the traders' preferences and endowments. The market owes its existence to the diversity of those who make up the economy.

An excess of diversity, however, could stretch the ability of economic institutions to operate efficiently: this has recently been a concern in regions experiencing extensive and rapid migration, such as the United States and the former Soviet Union. Are there natural limits on the degree of social diversity with which existing institutions can cope? This paper will argue that there are. I shall argue that not only is a certain amount of diversity essential for the functioning of markets, but at the other extreme, that too much diversity of a society's preferences and endowments may hinder its ability to allocate resources efficiently. This will be examined rigorously in the context of two classical forms of resource allocation: by markets and by social choice or voting, arguably those most frequently used in modern economies.

Until quite recently, diversity has been an elusive concept. However, a precise measure of *social diversity* will be given here in terms of the preferences and endowments of individuals. This concept is robust to small errors in measurements and is independent of the units of measurement. I shall establish that too much social diversity in this sense can interfere with the efficient performance of markets and with the achievement of social choices.

Shifting the angle of inquiry slightly sheds a different light on the subject. If a society allocates resources efficiently, whether by markets or by collective choices, then this society must exhibit no more than a certain degree of social diversity. There is therefore an implicit prediction here about the characteristics of economies that evolve mechanisms for allocating resources efficiently: they will have only a limited degree of social diversity in my sense.

The precise degree of social diversity that is consistent with the market reaching efficient allocations is described here by a condition of *limited arbitrage*. Intuitively, this gauges the extent of the gains from trade. This paper defines limited arbitrage precisely from endowments and preferences and then defines the degree of social diversity which it implies. It shows by means of examples why limited arbitrage separates those markets that have a competitive equilibrium from those that do not, and why it simultaneously separates those economies that have well-defined social choice rules from those that do not.

From this analysis a unified perspective emerges on the central question of resource allocation. This is the existence of a well-defined connection between two classic forms of resource allocation which have been considered separate and almost antagonistic until now: markets and public choices. The same limitation on social diversity links these two forms of resource allocation. Limited arbitrage is necessary and sufficient for the existence of competitive market allocations. It is also necessary and sufficient for the existence of well-defined social-choice rules. One can actually translate one form of resource allocation into the other. The success of both hinges on the same limitation on the social diversity of the economy. The economies

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which are observed in practice, if successful at either form of resource allocation, will exhibit a limited amount of social diversity. Turning this proposition around, it implies that increases in diversity beyond this threshold may call for forms of resource allocation different from both of the classic forms.

I. Limited Arbitrage and Gains from Trade

An economy E has $H \geq 2$ traders and $N \geq 2$ commodities or assets: the trading space is $X = \mathbb{R}^N$; without short sales $X = \mathbb{R}^{N+}$. Each trader has an endowment $\Omega_i \in \mathbb{R}^N$, and a preference represented by u_i: $\mathbb{R}^N \to \mathbb{R}$, appropriately normalized.[1]

The existence of both competitive equilibrium and social-choice rules is shown below to depend on the relation between the traders' *market cones*. These also provide a framework for measuring social diversity. One defines the ith traders market cone as those prices at which all trading opportunities which yield unbounded utility gains are unaffordable: $D_i = \{p \in \mathbb{R}^{N+}: \forall y \in A_i, \langle p, (y - \Omega_i)\rangle > 0\}$, where A_i consists of net trades with which the ith trader can achieve unbounded utility increases $A_i = \{y \in \mathbb{R}^N: \forall \lambda > 0, \ u_i(\Omega_i + \lambda y) > u_i(\Omega_i)$ and $\lim_{\lambda \to \infty} u_i(\Omega_i + \lambda y) = \infty\}$; A_i is called a *global cone*.[2] The economy has *limited arbitrage* when there exists one price at which no trader can afford unbounded utility increases.

Definition 1: The market economy E has *limited arbitrage* when all its market cones intersect: $\bigcap_{i=1}^{H} D_i \neq \varnothing$.

These concepts were introduced in Chichilnisky (1991, 1993b, c). The geometry of limited arbitrage is simple; it means that the traders' global cones A_i cannot contain net trades which add up to zero: all global cones A_i must lie on one side of some price hyperplane. Limited arbitrage relates to gains from trade, measured as the maximum increment in the sum of utilities which the traders can achieve by reallocating resources:

gains from trade

$$= G(E) = \sup\left(\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i)\right)$$

where for all i, $u_i(x_i) \geq u_i(\Omega_i)$ and $\sum_{i=1}^{H}(x_i - \Omega_i) = 0$.

PROPOSITION 1: *An economy E satisfies limited arbitrage if and only if it has bounded gains from trade, namely, $G(E) < \infty$.*

For a proof see Chichilnisky (1991, 1994).

Figure 1 illustrates an economy E_1 with two traders and two assets which has limited arbitrage. Its global cones are A_1 and A_2 and the price line p leaves both on one side. Net trade directions which lead to unbounded utility gains are unaffordable by all traders from their initial endowments at price p. Gains from trade in this economy $G(E_1)$ are bounded.

The economy of Figure 2 does not satisfy limited arbitrage: there are two directions of net trades $W_1' \in A_1$ and $W_1 \in A_2$, yielding unbounded increases in utility, and which sum up to zero. Therefore, there is no price p at which all net trades in A_1 and in A_2 are unaffordable from initial endowments. Gains from trade in this economy are unbounded.

Section III shows that the boundedness of possible gains from trade, which is now known to be equivalent to limited arbitrage, is fundamental to the existence of a competitive equilibrium: it is necessary and sufficient. Intuitively this is reasonable: an economy such as that in Figure 2, where traders wish to take unboundedly large and opposed trading positions, cannot reach an

[1] All the concepts and results in this paper are ordinal: they do not depend on the utility representations of the traders' preferences. Therefore, without loss of generality, I assume $\sup_{(x \in \mathbb{R}^N)} u_i(x) = \infty$. Throughout, $\langle x, y\rangle$ denotes the inner product between x and y.

[2] This definition is with short sales, $X = \mathbb{R}^N$. Without short sales, $X = \mathbb{R}^{N+}$, the *market cone* ∂D_i is $\partial D_i = D_i \cap S(E)$ if $S(E) \subset N$, and $\partial D_i = D_i$ otherwise, where $N = \{v \in \mathbb{R}^N: \exists i$ with $\langle v, \Omega_i\rangle = 0\}$, and where $S(E) = \{v \in \mathbb{R}^N: \exists(x_1 \dots x_H) \in \mathbb{R}^{H \times N+}$ with $\Sigma(x_i - \Omega_i) = 0, u_i(x_i) \geq u_i(\Omega_i)$ for all i, and $\forall z_i \in \mathbb{R}^{N+}, u_i(z_i) \geq u_i(x_i) \Rightarrow \langle v, z_i - x_i\rangle \geq 0\}$ (see Chichilnisky, 1991).

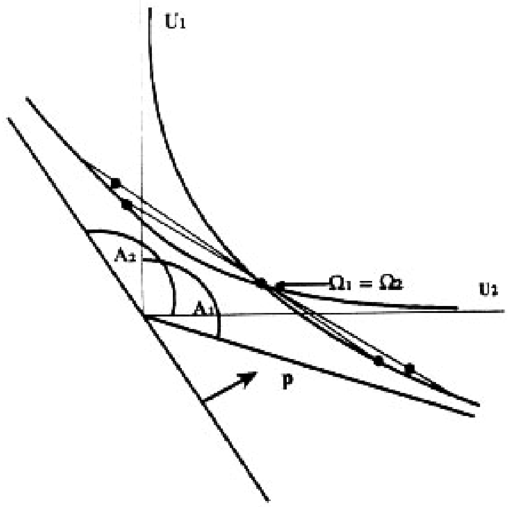

FIGURE 1. LIMITED ARBITRAGE IS SATISFIED

Notes: The two global cones lie in the half-space defined by p. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and such pairs of points lead to utility values below those of the endowments beyond a given distance from the initial endowments.

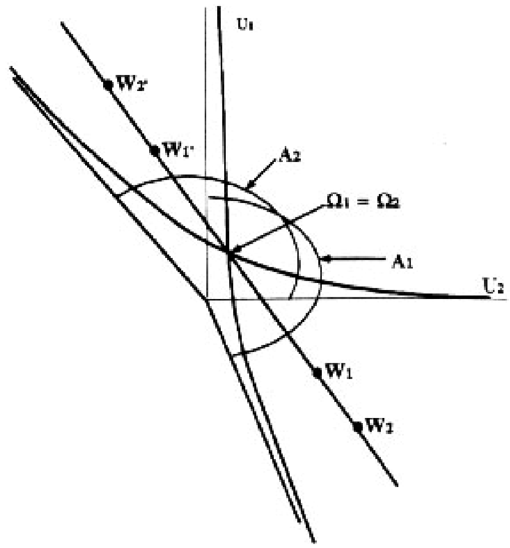

FIGURE 2. LIMITED ARBITRAGE DOES NOT HOLD

Notes: The global cones are not contained in a half-space, and there are sequences of feasible allocations such as (W_1, W_1'), (W_2, W_2') which lead to unbounded utilities.

equilibrium. Desired trades are just too diverse to be accommodated within the same economy.

II. Limited Arbitrage and No Arbitrage

In financial markets an *arbitrage opportunity* is the possibility of making gains at no cost, or equivalently, by taking no risks. *No arbitrage* means that such opportunities do not exist. It provides a standard way of pricing a financial asset, precisely so that no arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market-clearing equilibrium there is no arbitrage.

The link between limited arbitrage and no arbitrage is clear in an economy E with zero endowments, $\Omega_i = 0$ for $i = 1,2$. No arbitrage at the initial endowments means that no trade can increase the traders' utility at zero cost: gains from trade in E must therefore be zero. By contrast, E has *limited arbitrage* when no trader can increase utility beyond a given bound at zero cost: gains from trade are bounded, as seen in Proposition 1. No arbitrage requires *no* gains from trade at zero cost. Limited arbitrage requires that only bounded or *limited* gains can be achieved from trade.

The two concepts are related but nonetheless quite different. No arbitrage describes an allocation at which there is no further reason to trade. If applied at the initial allocations it means that the economy is autarchic, and not very interesting. Instead, limited arbitrage applies to the economy's initial data, the traders' endowments and preferences. It does not imply that the economy is autarchic: quite to the contrary, it is valuable in predicting whether the economy can ever reach a competitive equilibrium, by examining the economy's initial data.

III. Limited Arbitrage and Market Equilibrium

Limited arbitrage identifies fully those economies which have a competitive market equilibrium.[3] I consider competitive market allocations because they are Pareto efficient, which makes them desirable for resource allocation; other lesser concepts of equilibrium are not.[4] A competitive equilibrium is a price p^* and an allocation $x_1^* \ldots x_H^* \in \mathbb{R}^{N \times H}$ such that each trader $i = 1, \ldots, H$ maximizes utility within a budget, $u_i(x_i^*) = \max[u_i(x)]$ for $x \in \{y \in \mathbb{R}^N: \langle p^*, y - \Omega_i \rangle = 0\}$, and all markets clear, $\sum_{i=1}^{H}(x_i^* - \Omega_i) = \{0\}$. It was established in Chichilnisky (1991) that with[5] or without[6] short sales the following proposition holds:

PROPOSITION 2: *The economy E has a competitive equilibrium if and only if it satisfies limited arbitrage.*[7]

The proof is in Chichilnisky (1991).

Limited arbitrage is the first necessary and sufficient condition for existence of a competitive equilibrium in economies with or without short sales, with finite or infinite commodities (Chichilnisky, 1991; Chichilnisky and Geoffrey Heal, 1992). Other sufficient conditions in finite or infinite dimensions are in Chichilnisky and Heal (1993). It is intuitively clear that limited arbitrage is needed for an equilibrium to exist: otherwise traders with very diverse preferences wish to take unboundedly short and long positions against each other (see Figure 2). Desired trades are too diverse to be accommodated within the same economy.

It may seem surprising that limited arbitrage is also necessary for the existence of a competitive equilibrium in economies without short sales ($X = \mathbb{R}^{N+}$). However, the failure is the same with or without short sales: equilibrium fails when traders wish to take unboundedly large positions which the bounded resources of the economy cannot accommodate. Paradoxically, without short sales this occurs when some of the traders have zero income. Arrow and Hahn (1971) constructed a standard economy with no competitive equilibrium: one with two traders and two goods, without short sales, and in which preferences are continuous, concave and increasing. Trader 1 owns only the first good, which trader 2 does not like.[8] The second trader has strictly positive endowments. At an equilibrium the second trader does not trade, the first good is free, and the first trader has zero income. Trader 1 likes the first good, which is free.[9] Therefore there can be no competitive equilibrium: no allocation can maximize the first trader's utility when the first good is free. This example can be extended to economies with any number of traders and of goods, and in which some traders with positive income wish and can afford unbounded positions. Arrow and also Lionel McKenzie (1959) introduced *resource relatedness* and *irreducibility* to solve the nonexistence problem: their conditions ensure that all traders have nonzero income by restricting divergences in tastes and in endowments. They are sufficient but not necessary for existence

[3]The traders' preferences must satisfy $u_i: X \to \mathbb{R}$ is continuous, concave, $x \geq y = u_i(x) \geq u_i(y)$, $u_i(0) = 0$, and mild regularity conditions in footnotes 6 and 7. Cobb-Douglas, constant-elasticity-of-substitution, strictly concave, linear preferences and preferences with indifferences which intersect the axis are included.

[4]Quasi-equilibrium and compensated equilibrium are not generally Pareto efficient (Kenneth Arrow and Frank Hahn, 1971).

[5]With short sales the trading space is $X = \mathbb{R}^N$, assume that preferences are smooth (C^2), $\exists \epsilon, K > 0$: $\forall x \in \mathbb{R}^N$, $\|Du(x)\| > \epsilon$, and $\|D^2u(x)\| < K$, and the set of directions of gradients of an indifference surface which is not bounded below is closed.

[6]Without short sales $X = \mathbb{R}^{N+}$; assume that if an indifference surface of positive utility intersects the boundary of $\mathbb{R}^{N+}$ so do all indifference surfaces of higher utility.

[7]Without bounds on short sales, trades are in the positive orthant $\mathbb{R}^{N+}$, and the market cones are slightly different, denoted ∂D_i (see footnote 2). Limited arbitrage always means the same: all market cones intersect, $\bigcap_{i=1}^{H} \partial D_i \neq \emptyset$.

[8]That is, $\forall a > 0$ and $\forall x, y \in \mathbb{R}^2$, $u_2(x + a, y) = u_2(x, y)$.

[9]This is, $\forall x, y \in \mathbb{R}^2$ and $\forall a > 0$, $u_1(x + a, y) > u_1(x, y)$.

since an equilibrium can exist even though some traders have zero income. It should be obvious by now why the failure of existence is the same in economies with or without short sales. The failure originates from some traders wishing to take unbounded positions which a bounded economy cannot accommodate. An interesting angle on this problem is that, without short sales, the failure of existence occurs when some traders have zero income because what they own is of no market value.

It turns out that the existence of a competitive equilibrium is decided within sets of at most $N+1$ traders. In an economy E with H traders, each subset of traders $\theta \subset \{1 \dots H\}$ defines a *subeconomy* E_θ of E.

PROPOSITION 3: *The economy E has a competitive equilibrium if and only if every subeconomy E_θ with at most $N+1$ traders does, where N is the dimension of the trading space.*

The proof is in Chichilnisky (1991, 1993c).

The easiest way to visualize the connection between limited arbitrage and the existence of an equilibrium is in an economy E with two traders with linear utility functions[10] and with short sales. Such an economy has a competitive equilibrium when and only when the two traders' preferences are identical; otherwise it is always possible to find a sequence of affordable trades along which the utility of both traders increases without bound, such as that illustrated in Figure 2. This economy E has limited arbitrage precisely when the two linear preferences are identical, and only then: the global cones A_1 and A_2 of linear preferences are half-spaces, and the market cones D_1 and D_2 are half lines defined by the preferences' gradients. The market cones intersect if and only if the two gradients are identical. Therefore this economy has a competitive equilibrium if and only if it satisfies limited arbitrage.

[10]Utilities are linear when $\forall \alpha, \beta \in \mathbb{R}$, $u_i(\alpha x + \beta y) = \alpha u_i(x) + \beta u_i(y)$.

Simple nonlinear examples can also be given, such as that of Figure 2, which illustrates an economy without limited arbitrage. At any prices there is a sequence of affordable trades $(W_n, W_n')_{n=1,2\dots}$ in Figure 2, along which the traders can achieve unbounded utility levels, so that no competitive equilibrium can exist.

There are standard sufficient conditions that ensure the existence of a competitive equilibrium, such that no indifference surface intersects the boundary of $\mathbb{R}^{N+}$, or that all traders have a strictly positive endowment of every single good. These are very strong conditions, and Arrow and Hahn (1971 p. 80) find them "unrealistic." In any case all these conditions imply limited arbitrage, because being necessary for existence, limited arbitrage must be satisfied by any economy that has a competitive equilibrium.

IV. Social Diversity, Limited Arbitrage, and Efficient Markets

What if the economy does not have limited arbitrage? Then it is *socially diverse*.

Definition 2: The economy E is *socially diverse* when $\bigcap_{i=1}^{H} D_i = \emptyset$.

This concept is robust under small errors in measurement and is independent of the units of measurement or choice of numeraire (Chichilnisky, 1993c). If E is not socially diverse, all economies sufficiently close in endowments and preferences have the same property: the concept is *structurally stable*. Social diversity admits different "shades"; these can be measured, for example, by the smallest number of market cones which do not intersect:

Definition 3: The economy E has index of diversity $I(E) = H - K$ if $K+1$ is the smallest number such that there exists a subset of traders $\Upsilon \subset \{1 \dots H\}$ with cardinality of $\Upsilon = K+1$, and $\bigcap_{i \in \Upsilon} D_i = \emptyset$.

The index $I(E)$ ranges between 0 and $H-1$: the larger the index, the larger the social diversity. The index is smallest when all the market cones intersect: then all social diversity disappears and is replaced by limited arbitrage. Proposition 2 implies:

PROPOSITION 4: *The index of social diversity is $I(E)$ if and only if every subeconomy of E with $H - I(E)$ traders has a competitive equilibrium.*

V. Limited Arbitrage and Social Choice

It turns out that limited arbitrage, or the absence of social diversity, is also crucial for achieving resource allocation via social choice. Social-choice rules allocate resources by assigning a social preference $\Phi(u_1 \dots u_H)$ to each profile $(u_1 \dots u_H)$ of individual preferences[11] of an economy E, in a way which respects ethical axioms. The social preference ranks allocations in $\mathbb{R}^{N \times H}$, and is used to locate an optimal allocation. This procedure requires, of course, that an appropriate social choice rule Φ should exist: the role of limited arbitrage is important because it ensures existence. This was demonstrated rigorously in Chichilnisky (1993b), and will be illustrated below.

There are two main approaches to social choice. One is Arrow's: his axioms of social choice require that the rules Φ be nondictatorial, independent of irrelevant alternatives, and satisfy a Pareto condition. A second approach requires instead, that the rule Φ be continuous, anonymous, and respect unanimity (Chichilnisky, 1980, 1982, 1993b). Though the two sets of axioms are quite different, I show below that limited arbitrage is nevertheless closely connected with both. The connection is provided by the equivalence between limited arbitrage and bounded gains from trade (see Proposition 1).

Arrow's (1963) impossibility theorem established that a social-choice rule Φ does not exist in general; the problem of social choice has no solution unless individual preferences are restricted. Duncan Black (1948) established that "single peakedness" of preferences is a sufficient restriction. Using different axioms, Chichilnisky (1980, 1982, 1993a,b) established that a social-choice rule Φ does not generally exist. Chichilnisky and Heal (1983) established for the first time a necessary and sufficient restriction for the resolution of the social-choice paradox: the *contractibility* of the space of preferences,[12] which can be interpreted as a limitation on preference diversity (Heal, 1983). In all cases, therefore, the problem of social choice is resolved by restricting the diversity of individual preferences.

I shall show next that the traders' preferences in the economy E satisfy limited arbitrage if and only if they contain no *Condorcet triples* of large utility values. Condorcet triples are building blocks of Arrow's impossibility theorem and are at the root of the social-choice problem. Thus limited arbitrage eliminates the source of Arrow's impossibility theorem for choices of large utility values.

Definition 4: A *Condorcet triple* is a collection of three preferences over a choice set Z, represented by utilities u_i: $Z \to \mathbb{R}$, $i = 1,2,3$, and three choices α, β, and γ within a feasible set $Y \subset Z$ such that $u_1(\alpha) > u_1(\beta) > u_1(\gamma)$, $u_2(\gamma) > u_2(\alpha) > u_2(\beta)$ and $u_3(\beta) > u_3(\gamma) > u_3(\alpha)$.

Within an economy E, the social-choice problem is about the choice of allocations: choices are therefore in $Z = \mathbb{R}^{N \times H}$. An allocation $(x_1 \dots x_H)$ is *feasible* if $\sum_i (x_i - \Omega_i) = 0$. Preferences over allocations are induced naturally by the traders' preferences over private consumption defined above: $u_i(x_1 \dots x_H) \geq u_i(y_i \dots y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

Definition 5: The economy E with preferences $\{u_1 \dots u_H\}$ has *Condorcet triples of size* k if for every three preferences $u_1^k, u_2^k, u_3^k \in$

[11] In E the traders' preferences are defined over private consumption, u_i: $\mathbb{R}^N \to \mathbb{R}$; however, they define automatically preferences over allocations in $\mathbb{R}^{N \times H}$: $u_i(x_1 \dots x_H) \geq u_i(y_1 \dots y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

[12] A space X is contractible when there exists a continuous map f: $X \times [0,1] \to X$ and $x_0 \in X$ such that $\forall x$, $f(x,0) = x$ and $f(x,1) = x_0$.

$\{u_1 \ldots u_H\}$ there exist three feasible allocations $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k)$, $\beta^k = (\beta_1^k, \beta_2^k, \beta_3^k)$ and $\gamma^k = (\gamma_1^k, \gamma_2^k, \gamma_3^k)$ which define a Condorcet triple, and such that each trader achieves at least a utility level k at each choice: $\min_{i=1,2,3}\{[u_i^k, (\alpha_i^k), u_i^k(\beta_i^k), u_i(\gamma_i^k)]\} > k$.

The following shows that limited arbitrage eliminates Condorcet triples on matters of great importance, namely on those with utility level approaching the common supremum of utilities, denoted $S = \sup_{\{x \in Z\}} u_i(x)$.[13] The result uses the fact established in Proposition 1 that limited arbitrage is equivalent to bounds on gains from trade:

PROPOSITION 5: *Let E be a market economy with no bounds on short sales and $H \geq 3$ traders. Then E has social diversity if and only if its traders' preferences have Condorcet triples of every size.*

Equivalently, E has limited arbitrage if and only if for some $k > 0$, the traders' preferences have no Condorcet triples of size exceeding $S - k$.

PROOF:

Let E have limited arbitrage. For each $k > 0$, let $(\alpha^k, \beta^k, \gamma^k) \in \mathbb{R}^{3 \times N \times H}$ and $u_1^k, u_2^k, u_3^k \subset \{u_1 \ldots u_H\}$ be a Condorcet triple of size k. Without loss assume that $\forall_i, \Omega_i = 0$, and choose a utility representation: $\forall i, \sup_{\{x: x \in \mathbb{R}^N\}} u_i(x) = \infty$. By construction the three allocations are feasible $\forall k$ [e.g., $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in \mathbb{R}^{N \times 3}$, $\sum_{i=1}^3(\alpha_1^k) = 0$], and $\lim_{k \to \infty}[\min_{i=1,2,3}(u_i(\alpha_i^i), u_i(\beta_i^k), u_i(\lambda_i^k))] = \infty$. There exist therefore three traders, call them 1, 2, and 3, and a corresponding sequence of allocations $(\theta^k)_{k=1,2\ldots} = (\theta_1^k, \theta_2^k, \theta_3^k)_{k=1,2\ldots}$ such that $\forall k$, $\sum_{i=1}^H \theta_i^k = 0$ and $\forall i = 1,2,3$, $\sup_{k \to \infty} u_i(\theta_i^k) = \infty$. This implies that E has unbounded gains from trade, which contradicts Proposition 1. Therefore E cannot have Condorcet triples of every size.

Conversely, if E has no limited arbitrage, for any three traders, called 1, 2, and 3, with preferences u_1, u_2, and u_3, there exist three different vectors in $\mathbb{R}^N$, $a_1 \in A_1, a_2 \in A_2$, $a_3 \in A_3$ which are part of a feasible allocation $(a_1, a_2, \ldots, a_H)$, $\sum_{i=1}^H a_i = 0$. For any integer $k > 0$, and small $\varepsilon > 0$ consider the vector $\Delta = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^{N-}$ and the following three allocations: $\alpha^k = (ka_1, ka_2 - 2\Delta, ka_3 + 2\Delta, ka_4, \ldots, ka_H)$, $\beta^k = (ka_1 - \Delta, ka_2, ka_3 + \Delta, ka_4, \ldots, ka_H)$ and $\gamma^k = (ka_1 - 2\Delta, ka_2 - \Delta, ka_3 + 3\Delta, ka_4, \ldots, ka_H)$; each allocation is feasible (e.g., $ka_1 + ka_2 - 2\Delta + ka_3 + 2\Delta + ka_4 + \cdots + ka_H = k(\sum_{i=1}^H a_i) = 0$). For each $k > 0$ the three allocations α^k, β^k, and γ^k and the three utilities u_1, u_2, and u_3 define a Condorcet triple of size $m(k)$, with $\lim_{k \to \infty} m(k) = \infty$.

Turning now from Arrow's approach to the second approach to social choice (Chichilnisky, 1993b, c), the link connecting markets with social choices is still very close but takes a different form: the contractibility of the space of preferences, which is necessary and sufficient for continuous anonymous rules which respect unanimity (Chichilnisky and Heal, 1983), is shown to be equivalent to limited arbitrage (Chichilnisky, 1993c). Therefore, limited arbitrage, or equivalently the lack of social diversity, is necessary and sufficient for resource allocation via social-choice rules. Formally: let P consist of all those preferences which are *similar* to those of the market economy E, in the sense that their gradients are in the intersection of the market cones of the traders (see Chichilnisky, 1993b, c). Intuitively, a preference is similar to that of trader i when it prefers those allocations which assign i a consumption vector which u_i prefers:

THEOREM 1: *A continuous anonymous social choice rule Φ: $P^k \to P$ which respects unanimity exists for every $k \geq 2$ if and only if the economy E has limited arbitrage.*

For a proof see Chichilnisky (1993b).

[13]Since all concepts here are ordinal, without loss of generality I can assume that for all i, j, $\sup_{(x \in Z)} u_i(x) = \sup_{(x \in Z)} u_j(x) = S$.

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