ON STRATEGIC CONTROL

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I. INTRODUCTION

The purpose of this note is to clarify and complement several comments on my work [Chichilnisky, 1980, 1982a], which appeared in this *Journal*: Baigent [1987], Nitzan [1989], and Baigent [1989]. I shall offer two new results. The first result constructs a continuous, anonymous social selection rule that respects unanimity and the proximity of preferences, for a large family of preferences (Theorem 1). This complements the result of Nitzan [1989] and Baigent's [1989] extension of it and is achieved by accepting, as they do, that social outcomes be optimal choices, rather than rankings of all possible choices or social preferences. This result is of interest because it appears to contradict the original impossibility theorem of Chichilnisky,[1] which establishes that, for the same family of preferences, social aggregation rules[2] with these properties do not exist (Theorem 2).

The second result in this note goes in the opposite direction from the first: it shows that when outcomes are social preferences, then respect for unanimity alone is strong enough to imply the existence of strategic dictators. In particular, anonymity is ruled out. This result complements the impossibility results in Chichilnisky [1982a] and Baigent [1987]. Strategic dictators are players who have full control of the outcome: they can secure any outcome they desire by selecting appropriate choices within a noncooperative game framework (Theorem 3). For simplicity of exposition, I consider here linear preferences defined on euclidean choice spaces.[3]

The crucial difference that leads to the opposing sets of results is the specification of what social outcomes ought to be. When social outcomes are choices, respect of unanimity is a weak condition. It is in fact consistent with anonymity (Theorem 1). This

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1. Chichilnisky [1982a], proves that it is impossible to construct social choice rules which are continuous and anonymous, and respect unanimity.

2. *Social aggregation rules* assign a *social preference*, which is a complete, transitive relation on XxX, to any list of individual preferences, or profile. This is the framework in Chichilnisky [1982a] and in Baigent [1987]. Nitzan [1989] and Baigent [1989] work instead with rules that assign a *socially optimal choice*, i.e., a vector in the choice space X, to each profile of individual preferences.

3. The results are also valid for more general preferences, but at the cost of more notation.

complements and clarifies the results of the comments [Nitzan, 1989; Baigent, 1989], which prove existence of anonymous rules respecting a certain form of unanimity and of the proximity of preferences, by taking optimal choices as outcomes. I show that these results do not depend on the criterion of proximity chosen: the comments chose a special version of proximity, while I use a standard one. In both cases one gets similar results.

When one requires, instead, that outcomes be social preferences or rankings of choices, as it is traditional in social choice, matters are quite different. Unanimity is now a strong condition. By itself, unanimity leads to the existence of strategic dictators, when proximity of preferences is respected (Theorem 3). This latter result is more in keeping with the original impossibility results in Chichilnisky [1980, 1982a] and also with their discrete version presented in Baigent [1987]. In particular, the existence of strategic dictators rules out anonymity. The existence of strategic dictators who can control noncooperative games through strategic choice was also discussed in Chichilnisky [1982b] and Chichilnisky and Heal [1984].

II. Unanimity and Anonymity In Two Frameworks

There are k individuals. The choice space X is a closed convex subset of euclidean space. For simplicity, X is assumed to be a unit ball in R^n. A *choice* is therefore a point x in $X = \{(x_1, \ldots, x_n)$ such that $\Sigma x_i^2 \leq 1\}$. We consider as in Chichilnisky [1982a] linear preferences on X. Being linear, each preference is uniquely defined by a vector of unit length in R^n, the unit normal to its indifference surfaces. Therefore, the space of linear preferences P is identical to the unit sphere in R^n, $P = \{(\alpha_1, \ldots, \alpha_n): \Sigma\, \alpha_i^2 = 1\}$. Proximity of preferences is defined by the proximity of the vectors representing them, in the usual euclidean metric. The product space P^k, which is contained in P^{nk}, is provided with the standard euclidean metric in R^{nk}. Similarly, *proximity of choices* is defined by the standard metric on euclidean space R^n, which contains the choice space X. An *optimal social selection* φ is a function that assigns to each list of individual preferences $(p_1, \ldots, p_k) \in P^k$ an optimal choice $\varphi(p_1, \ldots,)p_k) = x$ in X. An optimal social selection is said to *respect the proximity of preferences* when it is a continuous function from P^k to X with their standard metrics. *A social aggregation* rule is a map from individual preferences to social preferences; i.e., a map $\tau: P^k \to P$. A social aggregation rule *respects unanimity* if for all p in

P, $\tau(p, \ldots, p) = p$. This means that if all individuals have exactly the same preferences or rankings of choices, then so does the social preference (see Chichilnisky [1982a] and Baigent [1987]). An optimal social selection φ *respects unanimity* if whenever all preferences in a profile share the same optimal choice x in X, then the social optimum assigned by the selection rule is also x. This is akin to the definition in Nitzan [1989] and Baigent [1989].

The difference between social aggregation rules and optimal social selections is clear. The former seek a complete social ranking of choices as an outcome, a (complete, transitive) social preference on X. The latter seek, instead, only one optimal social choice or selection, a vector in the choice space X. It is clear that the former is more demanding than the latter. The following result shows by how much.

Theorem 1 complements the existence results of Nitzan [1989] and Baigent [1989]. It establishes that anonymity and unanimity of optimal social selections are consistent properties in spaces of preferences in which, as shown in Chichilnisky [1982a] and Baigent [1987], social choice rules with the same properties cannot coexist.

THEOREM 1. There exists a continuous optimal social selection φ: $P^k \rightarrow R^n$ respecting unanimity and anonymity.

Proof of Theorem 1. For each profile of individual preferences $(p_1, \ldots, p_k) \in P^k$, let $\varphi(p_1, \ldots, p_k) = x \in X$, where

$$x = \Sigma 1/nx_i,$$

and, for all i, the choice x_i maximizes the preference p_i on X. This means that x_i maximizes the inner product with the vector p_i over X:

$$(p_i \cdot x_i) = \sum_j p_{ij}x_{ij} = \max_{y \in X} (p_i \cdot y).$$

The function φ is continuous, since it is the convex combination of the maxima of linear functions on a strictly convex set. It respects anonymity by construction. Unanimity is preserved because if all preferences have the same maximum x in X, the convex combination of x with itself is again the same optimal point x in X.

Q.E.D.

The following result contrasts the results obtained with the two different approaches:

THEOREM 2. There exists no social aggregation rule $\varphi: P^k \to P$ which is continuous and anonymous, and respects unanimity.

Proof of Theorem 2 See Chichilnisky [1980, 1982a]. For a discrete version see Baigent [1987].

Q.E.D.

III. STRATEGIC CONTROL

Consider now a social aggregation rule $\tau: P^k \to P$, which is continuous and respects unanimity. An *aggregation game G* is now defined as follows. There are k players. Each announces a preference, which is chosen strategically and may or may not be the true preference of the player. The institutional setup is one where only individuals know their true preferences. Each individual i has a ("true") preference p_i^T, and his/her strategy set consists of all the possible preferences in P. As each player announces a (strategically chosen) preference, a list $(p_1 \ldots, p_k) \in P^k$ is formed. This is the argument of τ. Player i evaluates the outcome $p = \tau(p_1, \ldots, p_k)$ with his/her true preferences p_i^T. The value of the outcome p to player i is given by the inner product $p_i^T p$, which is well defined since both p and p_i^T are vectors in the unit sphere of R^n: outcomes closer to p_i^T yield higher value. A player j is a *strategic dictator* if for all k-1 tuples of strategies played by the other players $(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$, there exists a strategy p_j^s for the j player such that

$$\tau(p_1, \ldots, p_{j-1}, p_j^s, p_{j+1}, \ldots, p_k) = p_j^T.$$

This means that by playing strategically, player j can obtain an outcome identical to his/her true preference, for all strategies played by the other players. Note that player j is not a dictator in general, since the optimal strategy p_j^s is usually different from p_j^T.

THEOREM 3. If the aggregation rule $\tau: P^k \to P$ is continuous and respects unanimity, then there exists a strategic dictator.

Proof of Theorem 3. The aggregation map τ defines a continuous map $\mu: (S^n)^k \to S^n$, since $P = S^n$. The unanimity condition implies that the index of this map μ restricted to the diagonal of $(S^n)^k = \{(z_1, \ldots, z_k): z_i \in S^n$, and for all $i, j, z_i = z_j\}$, has a topological degree equal to one (see Chichilnisky [1982a].[4] This implies that the degree of μ restricted to one of the copies of S^n within the product space $(S^n)^k$ is not zero. This is because if the degree of τ restricted to all these copies were zero, the degree on the diagonal would also be zero, a contradiction; see also Chichilnisky [1980, 1982a] and Chichilnisky and Heal [1984]. Without loss of generality, assume that the degree of τ is not zero on the first copy of S^n, denoted S^n_1. Then the first player is a strategic dictator. This is because for all k-1 strategies played by the other players, $(p_2, \ldots, p_k)$, the map τ restricted to the set $S(p_2, \ldots, p_k) = \{(p, p_2, \ldots, p_k):p \in S^n\}$, which is the set of all strategy lists that can be obtained by player 1 choosing all possible strategies open to him/her, and taking as given the other players's strategies, has also nonzero degree.[5] Therefore, the map τ restricted to $S(p_2, \ldots, p_k)$, $\tau/S(p_2, \ldots, p_k)$, is onto, i.e., it covers its image. Since this image is S^n, this implies that by an appropriate choice of strategy $p_j^s(p_2, \ldots, p_k) \in P$, player 1 can assure that the social aggregation rule τ will have as an outcome any preference in P and, in particular, his/her true preference p_i^T. Thus, player 1 is a strategic dictator. Note that the optimal strategy for player 1 may or not be p_i^T.

There may exist more than one strategic dictator, in which case the aggregation game has no Nash Equilibrium. However, if a Nash Equilibrium exists, then by definition the strategic dictator must maximize the outcome at this equilibrium, given the other players's strategies. This means that the outcome must be identical to player 1's preference.

Q.E.D.

COROLLARY 1. The outcome at a Nash Equilibrium of the aggregation game G is always identical to the preference of the strategic dictator whose existence Theorem 3 establishes.

Proof of Corollary 1. This follows directly from the proof of Theorem 3.

Q.E.D.

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4. The *topological degree* of a continuous function from the n-sphere S^n to itself, $f: S^n \rightarrow S^n$ is the number of times that the image of f "wraps around" S^n. This concept was defined in Chichilnisky [1982a] and was used to prove that unanimity implies that the degree of the social aggregation map restricted to the diagonal is one.

5. The set $S(p_2, \ldots, p_k)$ is a sphere and therefore the degree of τ restricted to this set is well defined. This degree is not zero, because the sphere S^n_1 is a continuous deformation of the sphere $S(p_2, \ldots, p_k)$ within $(S^n)^k$, and the degree of τ on S^n_1 is not zero by assumption.

REFERENCES

Baigent, N., "Preference Proximity and Anonymous Social Choice," *Quarterly Journal of Economics*, CII [1987], 162–69.

——, "Some Further Remarks on Preference Proximity," *Quarterly Journal of Economics*, CIV [1989], 191–94.

Chichilnisky, G., "Social Choice and the Topology of Spaces of Preferences," *Advances in Mathematics*, XXXVII [1980], 165–76.

——, "Social Aggregation Rules and Continuity," *Quarterly Journal of Economics*, XCVII [1982a], 337–52.

——, "The Topological Equivalence of the Pareto Condition and the Existence of a Dictator," *Journal of Mathematical Economics*, IX (1982b), 223–33.

Chichilnisky, G., and G. M. Heal, "Patterns of Power," *Journal of Public Economics*, XV [1984], 177–84.

Nitzan, S., "More on the Preservation of Preference Proximity and Anonymous Social Choice," *Quarterly Journal of Economics*, CIV [1989], 187–90.