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THE WALRASIAN MECHANISM FROM EQUAL DIVISION IS NOT MONOTONIC WITH RESPECT TO VARIATIONS IN THE NUMBER OF CONSUMERS

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1. Introduction

Perhaps the most widely advocated method of fairly dividing a bundle of infinitely divisible goods among a group of agents with equal claims on the goods is the method consisting of first dividing that bundle equally among the agents and then, operating the Walrasian mechanism. Indeed this method has a number of appealing features: it yields efficient allocations under standard assumptions on preferences; it does not involve interpersonal comparisons of utility; finally, it is compatible with many of the other equity criteria that have been proposed in the literature.

However, it has a serious limitation: if the number of agents increases while the resources at their disposal remain fixed, the bundle received by one of the original agents after account is taken of the presence of the new agents, may be strictly preferred by him to the bundle he had received at first. This is in violation of our moral intuition; a minimal amount of solidarity among agents should prevent that anybody gains if the claims of more agents have to be accommodated, while opportunities do not expand. The fact that the Walrasian mechanism operated from equal division does not satisfy this *population monotonicity* property – we will say that it is subject to the *population paradox* – was pointed out by Thomson (1981) in a

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paper where the property was formulated and studied in the context of the bargaining problem.

The question we address here concerns the extent of such violations. Can we impose natural restrictions on economies that would prevent them from occurring? In particular, can the violations occur in economies with a unique and stable Walrasian equilibrium?

The population paradox is related to the phenomenon known in international trade as the 'transfer paradox': it is sometimes possible for an agent to benefit from transferring to another agent part of his initial resources prior to the operation of the Walrasian mechanism. The transfer paradox had long been thought to be of little practical relevance because in the two-agent two-commodity case in which its analysis had been carried out, its occurrence required instability of the Walrasian equilibrium; however, it was recently shown by Chichilnisky (1980) that, as soon as the number of agents is equal to three, the paradox can occur in extremely 'well-behaved' economies, in particular in economies with a unique and stable Walrasian equilibrium.

The population paradox can be thought of as a transfer paradox by seeing the original economy, not as an economy with fewer agents than the final and enlarged economy, but as this enlarged economy in an initial position characterized by the fact that one of the agents has a zero endowment. However, the specification of the population paradox differs from the usual specification of the transfer paradox in that, (i) not only are the transfers from more than one agent, but also (ii) initial endowments are required to be identical across all agents with non-zero endowments both before and after the transfers, which of course implies, (iii) finiteness of the transfers (most of the literature on the transfer paradox concerns itself with marginal transfers).

If this interpretation is pursued, the following facts should be kept in mind: the population paradox cannot occur in two-person economies while the transfer paradox can. Second, in the case of more than two agents, three kinds of transfer paradoxes should be distinguished: the strong form of the paradox occurs when the donor gains and the recipient loses, and the weaker form when both donor and recipients lose, or when both donor and recipient gain. Clearly, the population paradox is not a transfer paradox of the first two kinds since the recipient, starting with zero consumption, can only gain from the transfer.

We establish here that the population paradox occurs frequently, and even in otherwise well behaved economies. In particular it can occur in economies with homothetic preferences and in economies with quasi-linear preferences. Under our assumption of equal endowments, economies with homothetic preferences have a unique and stable Walrasian equilibrium. Quasi-linear economies have a unique and stable interior equilibrium under the assumption of differentiability of preferences.

This note should be seen in the context of the considerable recent work on various paradoxes that affect the Walrasian mechanism, two other examples being the 'reallocation paradox' pointed out by Gale (1974), and the 'destruction paradox' discussed by Aumann and Peleg (1974). The former occurs when a group of agents achieve a Walrasian allocation that they all prefer by redistributing among themselves their initial endowments, the latter when an agent achieves a Walrasian allocation that he prefers by destroying part of his initial endowment. The frequent occurrence of these two paradoxes, documented in a number of studies, and of the population paradox, established here, indicates that very strong assumptions indeed are required to guarantee 'good' behavior of the Walrasian mechanism.

2. Notation; Definitions

There are l commodities. An *economy* is a triple $e=(n,(X_i,\succsim_i)_{i=1}^n,\Omega)$, where n is the number of agents, each agent i being characterized by a pair $(X_i,\succsim_i)$ of a consumption set $X_i\subseteq \boldsymbol{R}^l$ and of a preference relation $\succsim_i$ defined over X_i, and $\Omega\in\boldsymbol{R}^l_+$ is the aggregate endowment. A *feasible allocation for e* is a list $z=(z_i)_{i=1}^n\in\Pi X_i$ satisfying $\Sigma z_i=\Omega$. A feasible allocation for e, z, is a *Walrasian allocation from equal division* if there exists $p\in S^{l-1}$, the $(l-1)$-dimensional simplex, such that for each i, z_i maximizes $\succsim_i$ over agent i's budget set $B_i(p,n)=\{\bar{z}_i\in X_i|p\bar{z}_i\leqq p\Omega/n\}$. The set of all such z is denoted $W(e)$. The pair (p,z) is referred to as a *Walrasian equilibrium of e*. The economy $e=(n,(X_i,\succsim_i)_{i=1}^n,\Omega)$ *is a subeconomy of the economy* $e'=(n',(X_i',\succsim_i')_{i=1}^{n'},\Omega')$ if $n<n',(X_i,\succsim_i)=(X_i',\succsim_i')$ for all $i\leqq n$, and $\Omega=\Omega'$. The Walrasian mechanism operated from equal division exhibits the *population paradox at* (e,e') if e is a subeconomy of e', and there is an agent $i\in e$ such that for all $z\in W(e)$ and for all $z'\in W(e'),z_i'\succ_i z_i$.

3. Results

Theorem 1. The Walrasian mechanism operated from equal division may exhibit the population paradox in economies with homothetic preferences. In such economies, there is a unique and stable Walrasian equilibrium from equal division.

Proof. The proof of the first statement is by way of an example illustrated in fig. 1 (we will not give a formal algebraic description of the example, hoping that the illustration will be sufficiently convincing by itself). In the example, $l=2$ and we start with the economy $e=(2,(\boldsymbol{R}^2_+,\succsim_i)_{i=1}^2,\Omega)$, for which (p,z) with $z=(z_1,z_2)\in\boldsymbol{R}^{2\times 2}_+$ is a Walrasian equilibrium from equal division. Then we introduce a third agent, and in the enlarged economy $e'=(3,(\boldsymbol{R}^2_+,\succsim_i)_{i=1}^3,\Omega)$, the new Walrasian equilibrium from equal division

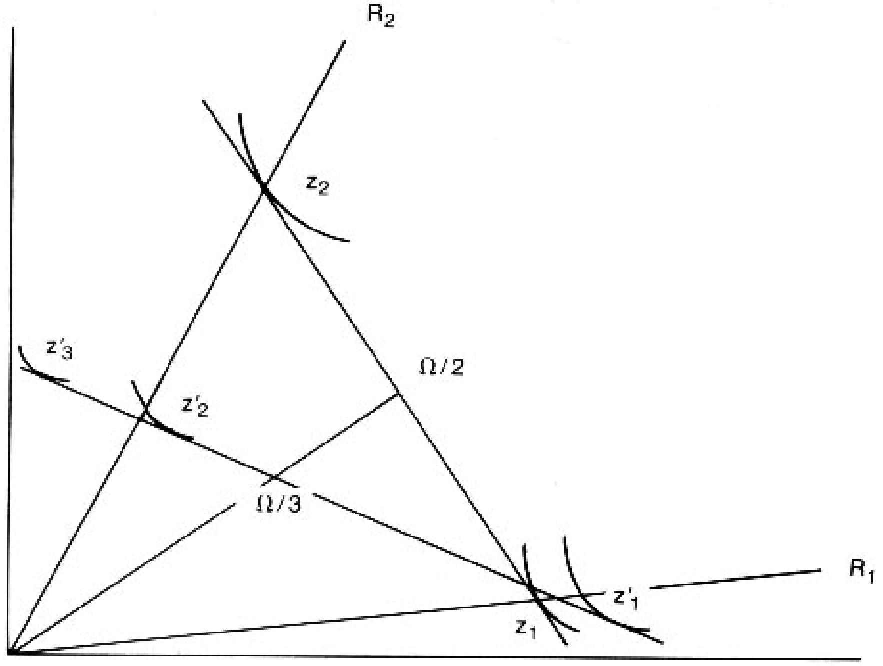

Fig. 1

(p', z') with $z' = (z'_1, z'_2, z'_3) \in \mathbf{R}_+^{2 \times 3}$ is such that $z'_1 \succ_1 z_1$. Agent 1 has benefitted from the arrival of the third agent, in violation of population monotonicity. Because z'_1 is below the ray R_1 through z_1, and z'_2 is below the ray R_2 through z_2, both agent 1 and agent 2's preferences can be chosen homothetic. Of course, agent 3's preferences can trivially be chosen to be homothetic since only one of his indifference curves is needed for the construction.

The fact that homotheticity of preferences and equality of initial endowments among consumers imply uniqueness and stability of Walrasian equilibria is a particular case of a result due to Eisenberg (1961). A detailed and direct proof appears in Chipman (1974). [Also, see Jones (1970)]. Q.E.D.

An intuitive explanation for the phenomenon described here is that the tastes of the new agent, agent 3, are so biased in favor of the good sold by agent 1 that the impact of agent 3's arrival on the price of that good more than compensates agent 1 for the decrease in his initial endowment.

It bears emphasizing that the assumption of homothetic preferences by itself is not sufficient to guarantee uniqueness of Walrasian equilibrium. Without the equal endowment assumption (actually, proportional endowments would have sufficed) uniqueness would not necessarily hold. The assumption of proportional endowments has been found to have strong regularizing implications for the behavior of aggregate demand [Chipman (1974)] and often has been imposed for analytical convenience. In our context, the stronger assumption of equal endowments is a natural one, which has important normative content.

Another class of economies that have been extensively studied and found convenient in a variety of contexts is the class of 'quasi-linear' economies, that is, economies in which all agents' preferences can be represented by utility functions that are separable additive in one good on the one hand and all the others on the other hand, and linear in the one good, i.e. functions $u_i: \boldsymbol{R}^l_+ \to \boldsymbol{R}$ such that $u_i(x_i, y_i) = x_i + v_i(y_i)$ for all $(x_i, y_i) \in \boldsymbol{R}^l_+$ for some $v_i: \boldsymbol{R}^{l-1}_+ \to \boldsymbol{R}$. (Note that this separability assumption implies that indifference curves are 'parallel': they can be obtained by horizontal translation of any one of them parallel to the x-axis and, consequently, all interior maximizing consumptions relative to the same prices have the same y component, i.e. income expansion paths are parallel to the x-axis in the interior of the consumption set.) Any such economy has an essentially unique and stable interior Walrasian allocation if preferences are differentiable. Also, the transfer paradox allocation if preferences are differentiable. Also, the transfer paradox cannot occur for transfers that do not disturb the interiority of the equilibrium allocations. Our final result is that the population paradox can occur even in quasi-linear economies. This result is, of course, not in contradiction with the observation concerning the transfer paradox, since, when the population paradox is interpreted as a transfer paradox in the way suggested in the introduction, the interiority assumption of the Walrasian allocation is obviously not met, one of the agents consuming nothing initially.

Theorem 2. Same as Theorem 1, except that the statement is made about the domain of quasi-linear economies.

Proof. The proof is by way of an example, with $l=2$, represented in fig. 2. Note that the two sample indifference curves drawn for each of agents 1 and 2 are 'parallel' to each other.

4. Concluding comments

In this note, we have argued that the population paradox may occur in extremely well behaved economies, just as the transfer paradox. Further results, by Jones (1984), indicate that in some classes of economies, the population paradox occurs in fact 'more frequently' than the transfer paradox.

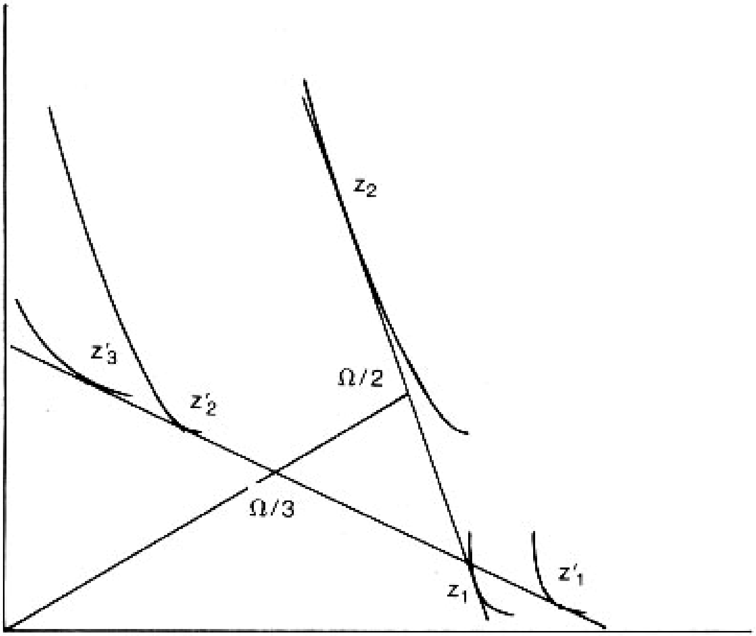

Fig. 2

References

Aumann, Robert, and Bezalel Peleg, 1974, A note on Gale's example, Journal of Mathematical Economics 1, 209–211.

Chipman, John, 1974, Homothetic preferences and aggregation, Journal of Economic Theory 8, 26–38.

Chichilnisky, Graciela, 1980, Basic goods, the effects of commodity transfers and the international economic order, Journal of Development Economics 7, 505–519.

Eisenberg, E., 1961, Aggregation of utility functions, Management Science 7, 337–350.

Gale, David, 1974, Exchange equilibrium and coalitions, Journal of Mathematical Economics 1, 63–66.

Jones, Ron, 1970, The transfer problem revisited, Economica, 178–184.

Jones, Ron, 1984, The population monotonicity property and the transfer paradox, University of Rochester, mimeo.

Thomson, William, 1980, The fair division of a fixed supply among a growing population, University of Rochester Discussion Paper.