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Competitive Equilibrium in Sobolev Spaces without Bounds on Short Sales*

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Following Chichilnisky and Chichilnisky-Kalman we establish existence and optimality of competitive equilibrium when commodity spaces are infinite dimensional Sobolev spaces, including Hilbert spaces such as weighted L_2 which have L_∞ as dense subspaces. We allow general consumption sets with or without lower bounds, thus including securities markets with infinitely many assets and unbounded short sales, and economies with production. We give non-arbitrage conditions on endowments and preferences which suffice for the existence of an equilibrium. Prices are in the same space as commodities. Equilibrium allocations are approximated by allocations in other frequently used spaces such as $C(R)$ and L_∞. © 1993 Academic Press, Inc.

1. INTRODUCTION

This paper establishes the existence and Pareto efficiency of competitive equilibrium in Arrow–Debreu exchange economies with infinitely many commodities and a finite number of consumers. It has two distinctive features. One concerns the assumptions on consumption sets: these are general convex sets which may include the whole space or be bounded below.[1] This feature allows us to prove existence of an equilibrium in markets without bounds on short sales, in both finite and infinite dimen-

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[1] The requirements on our consumption sets K are: K is closed and convex, and $y \geqslant x \in K \rightarrow y \in K$. This includes *inter alia*, the positive orthant, the whole space, and sets which are bounded below in some coordinates and not in others.

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sional cases. The ability to deal with the whole space as a consumption set is important where agents trade in contingent commodities, e.g., financial assets, and can naturally take unbounded short positions. Green [21], Hammond [22], Hart [24], and Werner [31] have all analysed equilibrium in particular finite dimensional economies where short sales are permitted, but their results do not apply to infinite dimensional cases, nor do they have the full generality required for Arrow–Debreu models.[2] We allow for consumption sets which are bounded or unbounded below, and mixed cases where some coordinates are bounded below and others not, thus covering a wide range of models of markets used in economics as well as in finance, and permitting an extension to economies with production. We handle unbounded consumption sets by introducing non-arbitrage assumptions on individual preferences (Condition (C)), and proving on the basis of these assumptions the norm-boundedness of the feasible allocations preferred by all agents to a given allocation. The interpretation of the results in securities markets with infinitely many assets is that non-arbitrage conditions suffice for the existence of a competitive equilibrium (Theorem 1). Related conditions, called *limited arbitrage*, have been shown to be necessary as well as sufficient for the existence of a competitive equilibrum in Euclidean and infinite dimensional commodity spaces, Chichilnisky [15, 16] and Chichilnisky and Heal [17].

The second distinctive feature is that commodities spaces are *Sobolev spaces*[3] H^S. These are Hilbert spaces of functions which include (weighted) L_2, and therefore have inner products and a countable orthonormal basis of coordinates. These are valuable features in the computation of solutions and in the study of dynamic models [8–10]: Sobolev spaces also provide a standard analytical framework for dynamics in physics (Adams [1]). Sobolev spaces have a further advantage: they can be made to consist exclusively of continuous and smooth functions, depending on the choice of the parameter s, as shown in the Appendix.[4] Sobolev spaces therefore allow the use of differential topology for the study of infinite dimensional problems (see Chichilnisky [7], Nirenberg [29]), thus providing in infinite dimensional cases the foundations for investigating properties of regular economies such as the number and stability of equilibria. In all cases, the

[2] Green, Hammond, Hart, and Werner deal with particular models; they typically consider two periods and incomplete markets.

[3] A definition of the Sobolev space H^s, where the integer $s \geqslant 1$, is given in the Appendix.

[4] For example $H^1 \subset C(R)$, the space of continuous real valued functions on R, $H^2 \subset C^1(R)$, the space of continuously differentiable real valued functions on R, and $H^0 = L_2$ (see the Appendix). Note that, as pointed out in [27] $C^1([0, 1])$ is not a Banach lattice, so results which require commodity spaces to be Banach lattices [27] cannot establish the existence of an equilibrium in C^1 spaces. However, since the Sobolev space H^2 consists of C^1 functions (see Appendix), our approach etablishes the existence of an equilibrium in spaces with commodities bundles in C^1.

duality properties of Sobolev spaces are good: prices, which are continuous linear functions on commodity spaces, are in the same space as commodities, thus avoiding the duality problems found in L_∞. The results presented here apply also to other L_p spaces, $1 \leqslant p < \infty$: this is shown in the Appendix. In the text we concentrate on the (weighted) Hilbert space[5] L_2. Hilbert spaces are the closest analog to Euclidean spaces in infinite dimensions, as they have a basis of coordinates and are self-dual. These spaces contain as dense subspaces other spaces which have been frequently used in economics, such as L_∞ (footnote 5) by Debreu [18] and subsequently *inter alia* Bewley [3] and Florenzano [20]. Hilbert spaces have been found attractive in non-parametric econometrics (Bergstrom [2]) and in the study of arbitrage in financial markets (Harrison and Kreps [23] and Chamberlin and Rothschild [6]). However, all Hilbert, indeed all the L_p spaces with $p < \infty$, present a well-known technical difficulty: the interior of the positive orthant is empty. This means that standard separation arguments used for finding equilibrium prices, such as the Hahn–Banach theorem, cannot be applied to sets contained in the positive orthant. This problem was solved several years ago in optimal growth models where Hilbert spaces were first introduced in the economics literature; Chichilnisky [8–10] and Chichilnisky and Kalman [12]. There are two main tools for dealing with this problem: one is to work with utility functions which are continuous in the norm of the space. This was done in Chichilnisky [8–10], where a complete characterization of such continuous functions was also provided (see the Appendix). The other tool is a "cone condition," which was introduced in Chichilnisky and Kalman [12, Theorem 2.1] as necessary and sufficient for the existence of supporting hyperplanes for convex sets which may have empty interiors. This condition was later adopted in Mas-Colell [27] and renamed "properness" (see Chichilnisky [14]). In addition to these tools, when consumption sets are unbounded below, we need here a Condition (C) on preferences. Condition (C) is used to prove that the feasible allocations exceeding given utility values for each agent form a bounded set even when the consumption set is all of H^s. Together with the continuity assumption in H^s this implies that the Pareto frontier is closed. Adding a regularity condition on supporting prices, we establish the existence and Pareto optimality of a competitive equilibrium in Theorems 1 and 2. The proofs in the text are for weighted L_2. The Appendix extends the results to weighted L_p $(1 \leqslant p < \infty)$ and to other Sobolev spaces of continuous and differentiable functions.

[5] A weighted L_2 space consists of measurable functions on R which are square integrable with respect to a finite measure μ on R, i.e., $\int \mu(t) < \infty$. L_∞, the space of uniformly bounded measurable functions on R, is contained in a weighted L_2 space as a dense subspace. Any element f of a weighted L_2 space is the limit in the $\|\cdot\|_2$ norm of a sequence of functions in L_∞, (f^n), $n = 1, ...,$ where $f^n(x) = f(x)$ for $|x| < n$, and $f^n(x) = 0$ otherwise.

The results presented here extend those in the literature in several directions. Our conditions for existence and optimality are given solely on individual preferences and endowments, as in the original proofs of Arrow and Debreu, rather than on derivative concepts such as demand. Most existence results consider only consumption sets which are positive orthants (Mas-Colell [27]), or more general sets (in R^∞)[6] but still bounded below (Boyd and McKenzie [4]); they exclude models frequently used in financial markets where consumption sets are unbounded below. Our consumption sets include a wider class of convex sets which contain the positive orthant, and which may be bounded below or not, or may even be bounded in some coordinates and not in others (see Sects. 5 and 6) and therefore include standard Arrow–Debreu models of markets with real commodities as well as models of financial markets. Moreover, for the special case of positive orthants our results are still more general.[7] For example, our conditions on preferences (continuity in H^s and regularity (R)) are strictly weaker than the conditions of [27] (see footnotes 4, 7, 10, and 13); [27] assumes an exogenous "closedness condition" on the Pareto frontier while we, instead, prove that the Pareto frontier is closed from our assumptions on preferences (see Lemma 5 and 7 and footnote 13); our results apply to consumption bundles in C^1 while those of [27] do not because C^1 is not a Banach lattice (see footnotes 4 and 7).

[6] R^∞ is the space of all sequences of real numbers.

[7] In Hilbert and L_p spaces, our existence results are more general than other existence theorems even in the case where consumption sets are positive orthants. Consider, e.g., Mas-Colell [27]. Our conditions are strictly weaker than those of [27] because we do not assume that the Pareto frontier is closed as in [27, "Closedness Hypothesis," p. 1046]. This condition is generally false in L_∞, as pointed out in [27], where it is postulated without reference to the primitives of the model (preferences and endowments). Our Lemmas 3, 5, 6, and 7 establish this property for the Pareto frontier and the existence of prices supporting Pareto efficient allocations from our assumptions on preferences; see footnote 13. Furthermore, the "uniform properness" condition of [27] is strictly stronger than our requirements on preferences (continuity); see footnotes 10 and 13. A "cone condition" identical to the "properness" condition of [27] was initially introduced in Chichilnisky and Kalman [12, Theorem 2.1, p. 25, (a)] and proven to be necessary and sufficient for the existence of a supporting hyperplane for convex sets which may have empty interior; Chichilnisky [14]. The results of [27] do not apply to spaces of C^1 functions because these are not Banach lattices; however, our H^2 spaces consist of C^1 functions (Appendix) so our results prove existence of C^1 equilibrium allocations. As this paper goes to press, Brown and Werner [5] circulated a new paper dealing with unbounded consumption sets which rules out the positive orthant in many spaces, while implying the "cone condition" (their condition A2). Therefore their results do not apply to Arrow–Debreu models in L_p, $1 \leqslant p < \infty$, but rather to models used in the finance literature. Brown and Wener [5] also require the "closedness condition" on the Pareto frontier while we do not.

2. Definitions

Commodities are indexed by the real numbers. *Consumption bundles* are therefore real valued functions[8] on R. The space H of commodity bundles is a weighted L_2 space of measurable functions $x(t)$ with the inner product $\langle x, y \rangle = \int_R x(t) \cdot y(t) d\mu(t)$, where $\mu(t)$ is any *finite* and *positive* measure on R ($\int_R d\mu(t) < \infty$, $\mu(A) > 0$) for all measurables in set A, which is absolutely continuous with respect to the Lebesgue measure. The L_2 norm of a function[9] x is $\|x\| = \langle x, x \rangle^{1/2}$. A *price* p is a real valued function on H giving positive value to positive consumption bundles: this implies that p is continuous on H, that p is itself a function in H, and that the value of the bundle x at price p is given by the inner product $\langle p, x \rangle = \int p(t) \cdot x(t)\, d\mu(t)$. The price space is therefore H^+, the positive cone of H. All results given here apply to the space of real sequences l_2 with a finite measure; the appendix extends the definitions and the results to other L_p and to Sobolev spaces H^s, for an integer $s \geqslant 1$.

The order $\geqslant$ in H is given by $x \geqslant y$ iff $x(t) \geqslant y(t)$ a.e., and $x > y$ iff $x \geqslant y$ and $x(t) > y(t)$ on a set of positive measure and $x \gg y$ iff $x(t) > y(t)$ a.e. A function $W: H \to R$ is *continuous* when it is continuous with respect to the norm of H. For all $x \in H$, define the set $W^x = \{y \in H : W(y) \geqslant W(x)\}$. A sequence (x^n) is said to converge to x in the weak topology iff $\langle x^n, h \rangle \to \langle x, h \rangle$ for all h in H. L_∞, the space of real valued functions on R which are bounded a.e., is a dense subset of H (footnote 5).

There are k agents, indexed by i. In all but the last section, the *consumption set* for the ith agent is H. In the last section we consider consumption sets K which are general convex sets in H. *Society's endowment* Ω is the sum $\Omega_1 + \cdots + \Omega_k$, where Ω_i is the initial non-negative endowment of individual i. A function u: $R^2 \to R$ is said to satisfy the *Caratheodory condition* if $u(c, t)$ is continuous with respect to $c \in R$ for almost all $t \in R$, and measurable with respect to t for all values of c. An allocation x is a vector $(x_1, ..., x_k) \in H^k$. A *feasible allocation* x is an allocation such that $\sum_i x_i \leqslant \Omega$. The *set of feasible allocations* is denoted F. Each individual i has a continuous real valued utility function W_i defined on a neighborhood of the consumption set K which is concave and increasing, i.e., if $u > v$, then $W_i(u) > W_i(v)$. The utility level of an allocation x denoted $W(x)$ is the k-dimensional vector $(W_1(x_1), ..., W_k(x_k))$, also called the *utility vector*. A utility vector is *weakly efficient* if there is no other feasible allocation $(z_1, ..., z_k)$ such that $W_i(z_i) \geqslant W_i(x_i)$ for all i, and $W_i(z_i) > W_i(x_i)$ for some i. The *Pareto frontier* is the set of weakly efficient utility vectors in the

[8] The analysis can easily be extended to real valued functions on R^n, $n > 1$.

[9] Chamberlin and Rothschild [6] give a straightforward economic interpretation of L_2 norms in models with infinite dimensional commodity spaces.

positive cone R^{k+}. Society's endowment Ω is said to be *desirable* if $W_i(\alpha\Omega) > W_i(0)$ for all $\alpha > 0$, and all i. This is always satisfied if W_i is strictly increasing, $W_i(0) = 0$, and the initial endowment Ω is positive. Let Δ denote the *unit simplex* in R^k, $\Delta = \{y \in R^{k+} : \sum_i y_i = 1\}$. A feasible allocation $(x_1, ..., x_k)$ is a *quasi-equilibrium* when threre is a price $p \neq 0$ with $\langle p, \Omega_i \rangle = \langle p, x_i \rangle$ and $\langle p, z \rangle \geqslant \langle p, x_i \rangle$ for any z with $W_i(z) \geqslant W_i(x_i)$, $i = 1, ..., k$. A feasible allocation $(x_1, ..., x_k)$ is an *equilibrium* when it is a quasi-equilibrium and $W_i(z) > W_i(x_i) \rightarrow \langle p, z \rangle > \langle p, x_i \rangle$. The latter holds at a quasi-equilibrium such that $\langle p, \Omega_i \rangle > 0$ for any i. The cone defined by a set Y and a $y \in Y$ is $C(Y, y) = \{z = \lambda(w - y) + y,\ w \in Y, \lambda \geqslant 0\}$.

3. One-Consumer Results

In this section and all others until Section 6, individual consumption sets are the whole space H. The following result addresses the problem of finding supporting prices for individually efficient positive commodity bundles (even though the positive cone of H has an empty interior). Because H is a Hilbert space and thus self-dual, a non-zero supporting price is always a non-zero function in H. Example 1 below shows *inter alia* that this result is not true in l_∞^+, even though l_∞^+ has an interior. The reason is that the dual of l_∞ contains "purely finitely additive measures"; these are non-zero continuous linear functions on l_∞ which do not admit representation by non-zero real valued functions.

Lemma 1. *Let $x \in H$ be a commodity bundle, and $W: H \rightarrow R$ a continuous, concave increasing function, for which there exists z with $W(z) > W(x)$. Then there exists a price $p \in H^+$ such that $\|p\| = 1$, and $\langle p, y \rangle \geqslant \langle p, x \rangle$ for all y satisfying $W(y) \geqslant W(x)$.*

Proof. This follows from the continuity of W; Chichilnisky [8, 9]. ∎

Example 1. *A continuous concave function on l_∞^+ which does not admit an extension to a continuous concave function on H, and provides a counter-example to Lemma 1 on l_∞^+.*

For $c \in [0, \infty)$, define $u_t(c) = 2^t c$ for $c \leqslant 1/2^{2t}$, $u_t(c) = 1/2^t$ for $c > 1/2^{2t}$ as shown in Fig. 1. For any sequence c in l_∞^+ let $W(c) = \sum_{t=1}^{\infty} u_t(c_t)$. Then if $\sup_t c_t < K$, $W(c) \leqslant K(\sum_t 1/2^t) < \infty$. W is thus well defined, continuous, concave, and increasing on l_∞^+. Let $a \in l_\infty^+$ be defined by $a_t = 1/2^{2t+1}$, and let $W^a = \{y \in l_\infty : W(y) \geqslant W(a)\}$. Now assume that p is a supporting price for the set W^a at a, i.e., p is a continuous positive linear function on l_∞ satisfying $p(y) \geqslant p(a)$ whenever $y \in W^a$. Let $p_t = p(e^t)$, where $e_j^t = 1$ if $t = j$, and 0 otherwise. These p_t define the "sequence part" of the continuous

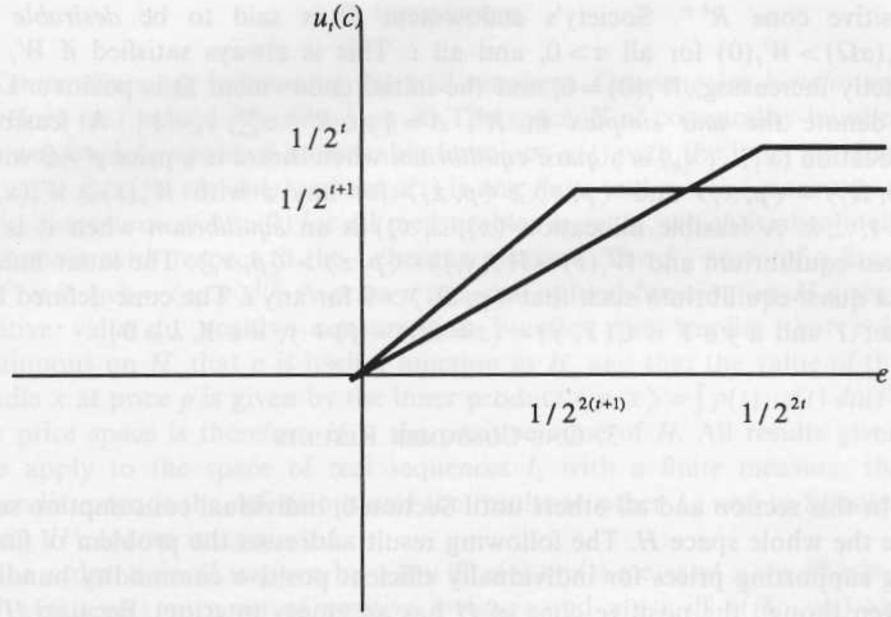

FIG. 1. Illustration of Example 1. The functon $W: l_\infty \to \mathfrak{R}$ is continuous and concave. It gives rise to preferred sets supported only by zero prices. This example shows that Lemma 1 cannot be obtained for functions on $l_{\infty+}$.

linear functional p. By the usual marginal rate of substitution arguments, $p_t = p_1 2^{t-1}$. We shall show that this leads to a contradiction when $p_1 \neq 0$. Define $z \in l_\infty^+$ by $z_t = 1/p_t$ and $z^n \in l_\infty$ by $z_t^n = z_t$ if $t \leqslant n$ and 0 otherwise. Then $z - z_n$ is nonnegative for all n, so that $p(z) \geqslant p(z_n)$. However, $p(z_n) = \sum_t p_t z_t = n > p(z)$ for some n sufficiently large, which is a contradiction. Therefore $p_1 = 0$ and $p_t = 0$ for all t, i.e., the sequence part of any supporting price p for W^a at a is identically zero. The only possible supporting prices for W^a are continuous linear functions on l_∞ whose sequence part is identically zero.

Note that not only is the function W continuous and concave on $l_\infty^+ \subset H$, but it is also well defined, continuous, and concave on H^+. However, the same argument shows that the set W^a cannot be supported at the allocation a by a non-zero price $p \in H$. It follows that W does not satisfy the conditions of Lemma 1. What fails is that W is not defined over all of H, and neither does it admit an extension to a continuous concave function defined over all of H: Lemma 2 shows why. The reason is that the slope of each $u_t(c)$ at 0 increases with t beyond any bound. For an extension of u to the negative orthant to be concave, the values of $u_t(c)$ when $c < 0$ must lie below the linear extension to negative c's of $u_t(c)$ for $0 \leqslant c \leqslant 1/2^{2t}$. Namely, for $c < 0$, $u_t(c) \leqslant 2^t c$. However, this would violate the necessary and sufficient conditions for continuity in H provided in Lemma 2 (see Fig. 2).

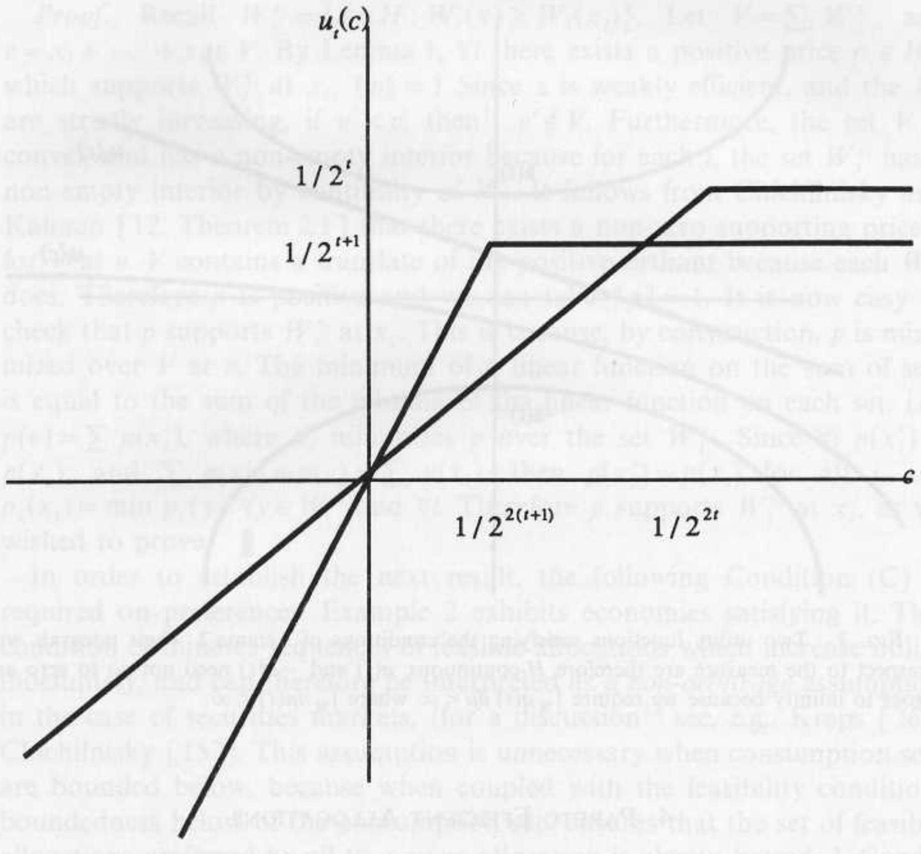

FIG. 2. A concave extension of W to the negative orthant contradicts the necessary and sufficient conditions for continuity in H given in Lemma 2, as illustrated in Fig. 3.

This shows that a continuous concave extension of W to all of H does not exist.

The space in which continuity is established is therefore important. For this reason we provide the following characterization of additively separable continuous functions on H:

LEMMA 2. *For some coordinate system of H, let* $W: H \to R$ *be defined by* $W(c) = \int_R u(c(t), t)\, d\mu(t)$, *where* $u: R^2 \to R$ *satisfies the Caratheodory condition. Then W defines a continuous function from H to R if and only if* $|u(c(t), t)| \leqslant a(t) + b\,|c|^2$, *where b is a positive constant,* $a(t) \geqslant 0$, *and* $\int_R a(t)\, d\mu(t) < \infty$.

Proof. See Chichilnisky [8]. ∎

Figure 3 illustrates continuity in H. An implication of Lemma 2 is that continuity in H implies a measure of discounting on the variable t. This point is developed in Heal [25].

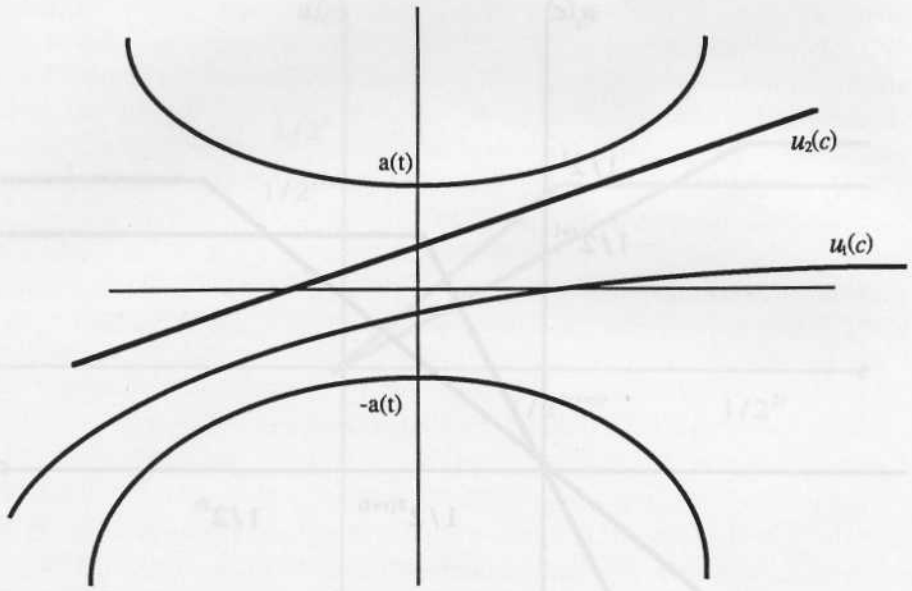

FIG. 3. Two utility functions satisfying the conditions of Lemma 2. Their integrals with respect to the measure are therefore H-continuous. $a(t)$ and $-a(t)$ need not go to zero as t goes to infinity because we require $\int_{\mathfrak{R}} a(t)\, d\mu < \infty$ where $\int_{\mathfrak{R}} d\mu(t) < \infty$.

4. Pareto Efficient Allocations

The next result extends Lemma 1 to the many agent case. Recall that in this section and the next, the individual consumption sets are all of H.

LEMMA 3. *Let $z = (x_1, ..., x_k)$ be a weakly efficient allocation, and for $1 \leqslant i \leqslant k$ let W_i be a concave, strictly increasing, continuous function. Then there exists a price $p \in H^+$, $\|p\| = 1$, such that $\langle p, y \rangle \geqslant \langle p, x \rangle$ for all y satisfying $W_i(y) \geqslant W_i(x_i)$.*[10]

[10] The parallel to our Lemma 3 in Mas-Colell [27] requires an additional condition denoted "uniform properness" which is not required here, and which is strictly stronger than our assumptions. We assume H^s continuity of the utilities instead, and it is easy to see that our condition of continuity implies "properness at one point" but it is strictly weaker than "uniform properness." To this end, consider a continuous strictly increasing function $W: H \to R$. Then $W(x - \lambda h) < W(x)$ implies that there is a ball around $(x - \lambda h)$ where this inequality is also satisfied. Therefore for all x there exists h in H^+ and a neighborhood V of the origin such that $W(x - \lambda h + y) > W(x)$ implies $y \notin \lambda V$. The "uniform properness" condition of [27] requires the existence of one $h \in H^+$ and a neighborhood V of $\{0\}$ which are valid for all x in H such that $W(x - \lambda h + y) > W(x)$ implies $y \notin \lambda V$. Our condition of continuity is therefore strictly weaker than the "uniform properness" condition of [27], since in our case the h and δ may vary with x, while "uniform properness" requires instead the existence of one h and one δ which must be the same for all x in H. It is also easy to see that properness at

Proof. Recall $W_i^{x_i} = \{x \in H : W_i(x) \geqslant W_i(x_i)\}$. Let $V = \sum_i W_i^{x_i}$, and $v = x_i + \cdots + x_k \in V$. By Lemma 1, $\forall i$ there exists a positive price $p_i \in H^+$ which supports $W_i^{x_i}$ at x_i, $\|p\| = 1$ Since z is weakly efficient, and the W_i are strictly increasing, if $v' < v$, then[11] $v' \notin V$. Furthermore, the set V is convex and has a non-empty interior because for each i, the set $W_i^{x_i}$ has a non-empty interior by continuity of W_i. It follows from Chichilnisky and Kalman [12, Theorem 2.1] that there exists a non-zero supporting price p for V at v. V contains a translate of the positive orthant because each $W_i^{x_i}$ does. Therefore p is positive and we can take $\|p\| = 1$. It is now easy to check that p supports $W_i^{x_i}$ at x_i. This is because, by construction, p is minimized over V at v. The minimum of a linear function on the sum of sets is equal to the sum of the minima of the linear function on each set, i.e., $p(v) = \sum p(x_i')$, where x_i' minimizes p over the set $W_i^{x_i}$. Since $\forall i$ $p(x_i') \leqslant p(x_i)$, and $\sum_i p(x_i') = p(v) = \sum_i p(x_i)$, then $p(x_i') = p(x_i)$ for all i, i.e., $p_i(x_i) = \min p_i(y)$ $\forall y \in W^{x_i}$ and $\forall i$. Therefore p supports $W_i^{x_i}$ at x_i, as we wished to prove. ∎

In order to establish the next result, the following Condition (C) is required on preferences. Example 2 exhibits economies satisfying it. This condition eleminates sequences of feasible allocations which increase utility indefinitely, and can therefore be interpreted as a *non-arbitrage* assumption in the case of securities markets, (for a discussion[12] see, e.g., Kreps [26], Chichilnisky [15]). This assumption is unnecessary when consumption sets are bounded below, because when coupled with the feasibility condition, boundedness below of the consumption sets ensures that the set of feasible allocations preferred by all to a given allocation is always bounded. Condition (C) is, however, needed to prove the boundedness of the feasible allocations preferred by all to a given allocation when consumption sets are

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one point is identical to the *cone condition* introduced first in Chichilnisky and Kalman [12, Theorem 2.1, p. 25] as a necessary and sufficient condition for the existence of a supporting hyperplane for convex sets which may have an empty interior; see also Chichilnisky [14]. The *cone condition* of Chichilnisky and Kalman [12] requires that if Y is a convex set, then there exists a w which is at a positive distance from the cone defined by Y and $x \in Y$, $C(Y, x) = \{z = a(y - x) + x,\ \ y \in Y,\ \ a \geqslant 0\}$; see Chichilnisky and Kalman [12, (a) of Theorem 2.1 p. 25]. When Y is the convex set of points which are preferred (by $\geqslant_i$) to x, this condition is identical to requiring that there exists a vector $w \geqslant 0$, and an open neighborhood of the origin V such that: $(x - \alpha w) + z \geqslant_i x$ implies $z \notin \alpha V$, which is the "properness" condition for the preference $\geqslant_i$ [27, p. 1043], see Chichilnisky [14].

[11] For otherwise there would exist an allocation $z'' = (x_1'', ..., x_k'')$ s.t. $\sum_i x_i'' \leqslant v$ so that z'' is feasible, and such that $W_i(x_i'') > W_i(x_i)$, contradicting the fact that z is weakly efficient.

[12] This relates to other non-arbitrage conditions, e.g., the limited arbitrage condition of Chichilnisky [15]: (C) implies (and in some cases it is equivalent to) the existence of a price $p > 0$, such that if a sequence of feasible allocations $(z + \Omega)_n = ((x_1)_n + \Omega_1, ..., (x_k)_n + \Omega_k)$ satisfies $\lim_n W^i((x_i)_n + \Omega_i) = \sup_{y \in X} W^i(y)$ for some i and $W^j((x_j)_n + \Omega_j) \geqslant W^j(\Omega_j)$ for all other j, then there exists N s.t. $\langle p, (x_i)_n \rangle > 0$ for all $n > N$.

not bounded below, as is the case in models with unbounded short sales where the consumption set is all of H. Condition (C) means that it is not possible to obtain unbounded utility from bounded resources.

(C) *Non-arbitrage.* Let $z=(x_1, ..., x_k) \in F$. If a sequence $(z^n) \in F$ satisfies $\|z^n\| \to \infty$ and $z_i^n \in W_i^{x_i}$, then $\exists\, N$ s.t. $(\Omega - z_i^n) \notin \sum_{j \neq i} W_j^{x_j}$ for $n > N$.

LEMMA 4. *Let $W_i \colon H \to R$ be increasing for all i. Condition* (C) *implies that the set of feasible allocations at which the utility values for each agent achieve or exceed those achieved at any given $z \in F$ is norm bounded.*

Proof. Let $z^n = (x_1^n, ..., x_k^n)$ be a sequence of feasible allocations in F (defined in Sect. 2) such that $\|z^n\| \to \infty$. Without loss of generality we may assume that $\sum_i x_i^n = \Omega$ for all n. By assumption (C), if for some i and all n, $x_i^n \in W_i^{x_i}$, then $\exists j$ s.t. $x_j^n \notin W_j^{x_j}$ for $n > N$, so that the set of feasible allocations in which each agent's utility level achieves or exceeds those achieved by each agent at the allocation z is norm bounded. ∎

EXAMPLE 2. *An economy satisying Condition* (C). Consider an economy with a social endowment $\Omega \in H^+$, and with two agents having the same continuous strictly concave and increasing utility $W \colon H \to R$, where $W(c) = \sum_t u(c(t))\mu(t)$, $W(0) = 0$, $u \colon R \to R$, $u(c) = c$ if $c < 0$, and $u(c) = \alpha c$ if $c \geqslant 0$, for some number α, $0 < \alpha < 1$. By Lemma 1, W is continuous on H. Note that u satisfies two properties. Property (a) is that $\forall a > 0$, $u(a+b) \geqslant u(a) + u(b)$. This is obvious when $b > 0$ because in this case, $u(a+b) = u(a) + u(b)$. Now take $b < 0$ and $(a+b) \geqslant 0$. Then $u(a+b) = \alpha(a+b) > \alpha a + b = u(a) + u(b)$, because $a > 0$ and $b < 0$. And when $b < 0$ and $(a+b) < 0$, then $u(a+b) = a + b > \alpha a + b = u(a) + u(b)$. Therefore $\forall a > 0$, $u(a+b) \geqslant u(a) + u(b)$. Property (a) of u implies that for all $w \in H^+$, $W(w+x) \geqslant W(w) + W(x)$. The function u satisfies a second property (b): $\forall c$, $\exists N > 0$, s.t. $u(c-v) - u(c) < -[u(c+v) - u(c)]$ if $\|v\| > N$. Property (b) of u implies that if (x^n) is a sequence in H, $\|x^n\| \to \infty$, and $W(x^n) > W(0) = 0$, then $\exists N$ s.t. for $n > N$ and all $w \in H^+$, $W(w - x^n) < W(w)$. We may now prove Condition (C) for this economy. Assume without loss of generality that the initial endowments of the agents are $0 \in H^+$ and $y \in H^+$, respectively. Since W is increasing we may also assume without loss that $y = \Omega$. Consider now an unbounded sequence of allocations $(z^n) = (x^n, y^n) \in H^2$. By monotonicity of W we may assume that $y^n = \Omega - x^n$, so that both x^n and y^n are unbounded. Therefore we consider an unbounded sequence (x^n) in H along which the utility of the first agent increases, i.e., $W(x^n) \geqslant W(0)$. We shall prove that the utility of the second agent will eventually decrease, so that Condition (C) is satisfied. Since $\|x^n\| \to \infty$, $W(x^n) \geqslant W(0) = 0$, and $\Omega \in H^+$, property (a) implies that along this sequence $W(\Omega + x^n) \geqslant W(\Omega)$ for n sufficiently large: this is because $W(\Omega + x^n) = \sum_t [u(\Omega(t) + x^n(t))]\mu(t)$, and for each t, by (a),

$u(\Omega(t) + x^n(t)) \geqslant u(\Omega(t)) + u(x^n(t))$ so that $W(\Omega + x^n) \geqslant \sum_t u(\Omega(t)) = W(\Omega)$. Finally since $W(\Omega + x^n) \geqslant W(\Omega)$ for large n, by property (b) $W(\Omega - x^n) < W(\Omega)$ for large n, as we wished to prove. It is tedious but standard to extend this example to $k \geqslant 2$ agents having similar but different utilities.

The next lemma establishes that the Pareto frontier is closed.[13]

LEMMA 5. *Assume that society's endowment Ω is desirable, and the utilities of the agents are continuous, strictly increasing, concave, and satisfy Condition* (C). *Then on any ray r of the positive cone in R^{k+} there exists a non-zero weakly efficient utility vector. The Pareto frontier is closed, and the map $v(r) = \sup_j(W^j)$, $\forall W^j \in r$, is continuous in r.*

Proof. Since the initial endowment is desirable and each W_i is increasing for each ray r in R^{k+} there exists a feasible allocation $(x_1^0, ..., x_k^0)$ s.t. $W_1(x_1^0), ..., W_k(x_k^0)$ is a non-zero vector in r. Consider the set $S = \{(x_1, ..., x_k) \in F : \forall i,\ W_i(x_i) \geqslant W_i(x_i^0)\}$. S is a bounded set by Lemma 4, and is convex and closed in H^k. Since H^k is reflexive, its weak topology coincides with its weak* topology: by the Banach–Alaoglu theorem (Dunford and Schwartz [19]) S is weakly compact. Consider now a sequence of utility vectors (W^j) contained in the ray $r \in R^{k+}$. Without loss of generality we may assume that (W^j) is increasing. By definition, $W^j = W_1(x_1^j), ..., (W_k(x_k^j))$ for some sequence $(x^j) = (x_1^j, ..., x_k^j) \in F$. Let $v(r) = \sup_j(W^j)$ in r. We shall prove that $v(r)$ is a utility vector corresponding to some feasible allocation. Let x be the weak limit of the sequence (x^j) in S, which exists because S is weakly compact. The Banach–Saks theorem (Dunford and Schwartz [19]) implies that there is a subsequence of (x^j), denoted also (x^j), such that $\lim_m((x^1 + \cdots + x^m)/m) = x$ in the norm. Therefore,

$$\lim_m \left(\sum W_i(x_i)/m\right) = \lim_j W_i(x_i^j) = v_i, \tag{1}$$

the ith component of $v(r) = \sup_j(W^j)$: this is the Banach–Saks theorem in

[13] This lemma cannot be established in l_∞ or L_∞, where it is generally false, exemplifying the problems of interpretation arising from the use of such spaces. An example in [27] illustrates this point by showing a Pareto frontier in l_∞ which is not closed even though all preferences are l_∞ continuous. To prove existence of an equilibrium, Mas-Colell [27] *assumes* that the Pareto frontier is closed (see [27, "closedness condition" on p. 1046]), an assumption which is made without reference to primitive conditions on preferences or endowments. A similar assumption appears in Brown and Werner [5, assuption A1]. Here, instead, we *prove* that the Pareto frontier is closed from the properties of preferences. This is done even when the consumption set for each consumer is the positive orthant as in [27], and it is also proved here where the consumption set of each agent is the whole space H, or more general consumption sets. The latter two cases are not considered in [27].

the line. Since S is convex, $(x^1 + \cdots + x^m)/m$ is in S for all m. By concavity of W_i,

$$W_i\left(\sum_j x_i^j/m\right) \geqslant \sum_j W_i(x_i^j)/m. \tag{2}$$

Note, however, that

$$W_i\left(\sum_j x_i^j/m\right) \leqslant W_i(x_i^m) \tag{3}$$

because by assumption the sequence of utility vectors (W^j) is increasing in the ray r and the function W_i is increasing and concave. Since S is bounded and W_i is continuous, (1), (2), and (3) imply that $\forall i = 1, ..., k$, the sequence $W_i((x_i^1 + \cdots + x_i^m)/m)$ converges to v_i in R^k. This means that $v(r) = \sup_j (W^j)$ is reached within S, for any sequence of utility vectors (W^j) in r. Since S is bounded and W_i is continuous, $\exists W(r)$ in r s.t. $W(r) \geqslant (W^j)\ \forall j$, for any sequence (W^j) in r. This completes the proof that along every ray r in R^k there is a non-zero weakly efficient vector. We shall now show that the Pareto frontier is closed in R^{k+}. If the limit $v \in R^{k+}$ of a sequence (W^j) in the Pareto frontier is not in this frontier, consider the ray r in R^{k+} through the origin, passing through this limit point v. There exists a non-zero weakly efficient utility vector $\mu \in R^{k+}$ on that ray. If $\mu \neq v$ then either μ is not weakly efficient or else for $j \geqslant N$, some N, W^j is not weakly efficient. In either case we have a contradiction. Thus $\mu = v$ and the Pareto frontier is closed. The last statement of the lemma is the closed-graph theorem. ∎

5. Existence and Optimality of Competitive Equilibrium with Unbounded Short Sales

The following regularity condition is now required on preferences. It ensures that there exist supporting hyperplanes to efficient allocations that do not approach $\{0\}$ weakly.

(R) There exist an agent i, $1 \leqslant i \leqslant k$, and a non-empty set $\Pi \subset H^+$ such that $0 \notin$ the weak closure of Π and $\Pi \subset$ the set of supporting hyperplanes to the preferred set W_i^x at x, $\forall x \in H$.

Example 3. *Preferences satisfying Condition* (R). Condition (R) is satisfied by a smooth preference W_i whose gradient vectors $DW_i(x)$ belong to a weakly closed set disjoint from the origin, or to a norm closed, norm bounded and convex set disjoint from the origin. Condition (R) is satisfied when there exists a vector v and an $\varepsilon > 0$ such that for all $x \in H$ the supporting hyperplanes for W_i^x have normals $p_i(x)$ with $\langle p_i(x), v \rangle > \varepsilon > 0$. This latter condition is satisfied for example by all preferences in [27]. The preferences of Example 2 satisfy Condition (R). In the following result, the agent's consumption set is H:

THEOREM 1. *Consider an economy with a desirable initial endowment* $\Omega = (\Omega_1, ..., \Omega_k)$ *in* H^{k+}, *and such that the individual utilities* $W_i: H \to R$ *are continuous, strictly increasing, concave, and satisfy Conditions* (C) *and* (R). *Then there exists a competitive quasi-equilibrium allocation* $(x_1^*, ..., x_k^*)$ *with a supporting price* $p \in H^x$, $\|p\| = 1$. *This allocation is a competitive equilibrium when all initial endowments* Ω_i *are strictly positive. The competitive equilibrium is Pareto efficient.*

Proof. We define a correspondence $\phi: \Delta \to T = \{y \in R^k : \sum_{i=1}^k y_i = 0\}$ with the property that any of its zeroes is a quasi-equilibrium.[14] For each $r \in \Delta$, let $x(r) = (x_1(r), ..., x_k(r))$ be the feasible allocation which gives the greatest utility vector colinear with r. Such an allocation defines a non-zero utility vector which depends continuously on r by Lemma 5. Without loss of generality assume that $\sum_i x_i(r) = \Omega$. Now let $P(r) = \{p \in H^+ : \|p\| \leqslant 1$, p supports $x(r)\}$. $P(r)$ is convex, and is non-empty by Lemma 3. Now define $\phi(r) = \{(\langle p, \Omega_1 - x_1(r)\rangle, ..., \langle p, \Omega_k - x_k(r)\rangle) : p \in P(r)\}$. $\phi(r)$ is non-empty and convex valued, $\sum_i z_i = 0$ for $z \in \phi(r)$ by Walras' Law, and $0 \in \phi(r)$ if and only if $x(r)$ is a quasi-equilibrium.

The next step is to show that ϕ is upper semicontinuous, i.e., if $r^n \to r$, $z^n \in \phi(r^n)$, $z^n \to z$ then $z \in \phi(r)$. Consider now the feasible allocation $x(r)$ in H^k, where $r = \lim_n(r^n)$. Let u be any other allocation with $W_i(u_i) > W_i(x_i(r))$, where $x_i(r)$ is the ith component of the vector $x(r)$. Since $r^n \to r$, eventually $W_i(u_i) > W_i(x_i(r^n))$, which implies $\langle p^n, u_i\rangle \geqslant \langle p^n, x_i(r^n)\rangle = \langle p^n, \Omega_i\rangle - z_i^n$, where z_i^n is the ith component of $z^n \in \phi(r)$, and $p^n \in P(r^n)$: this follows from the definitions of z^n and of p^n. Let (p^n) be any such sequence of price vectors in $P(r^n)$. Closed convex bounded sets in H are weakly compact by the Banach-Alaoglu theorem (Dunford and Schwartz [19]) because H is reflexive; thus the set $\{p: \|p\| \leqslant 1\}$ is weakly compact.[15] The weak closure of the set $\bigcup_r P(r)$ of supporting prices to the preferred sets of the agents is contained within $\{p : \|p\| \leqslant 1\}$, and is weakly compact as well. There exists therefore a p with $\|p\| \leqslant 1$ and a subsequence (p^m) of (p^n) such that $\langle p^m, f\rangle \to \langle p, f\rangle$ for all f in H. Note that by Lemma 3 each p^m supports the preferred sets of all agents at $x(r)$, so that by Condition (R) the weak limit of $p^m = p \neq 0$,

[14] This follows a method which was introduced by Negishi [28].

[15] In L_∞, the Banach–Alaoglu theorem proves that convex, bounded, and closed sets are instead weak* compact (Dunford and Schwartz [19]). Only in reflexive spaces is the weak* topology equal to the weak topology.

so we may take $\|p\|=1$. In particular such a p exists for $f=u$, i.e., $\langle p^m, u\rangle \to \langle p, u\rangle$. Therefore in the limit $\langle p, u\rangle \geqslant \langle p, \Omega_i\rangle - z_i$. Since this is true for all u with $W_i(u_i) > W_i(x_i(r))$, it is also true for u with $W_i(u_i) \geqslant W_i(x_i(r))$ and in particular for $u=x$, i.e., $\langle p, x_i(r)\rangle \geqslant \langle p, \Omega_i\rangle - z_i$ for all i so that $\langle p, \sum_i x_i(r)\rangle \geqslant \langle p, \sum_i \Omega_i\rangle - \sum_i z_i$. Since $\sum_i x_i(r) = \sum_i \Omega_i$ and $\sum_i z_i = 0$, we have $\langle p, x_i(r)\rangle = \langle p, \Omega_i\rangle - z_i$ for all i, implying that $z \in \phi(r)$ as we wanted to prove. The proof is completed by showing that ϕ has a zero. This is now a standard application of Kakutani's fixed point theorem. Consider the map Γ defined by $\Gamma(r) = r + \phi(r)$. It is upper semicontinuous, non-empty, and convex valued. If r is in the boundary of Δ, $\Gamma(r) \in \Delta$: if $r_i = 0$ for some i, then $x_i(r)$ is indifferent to 0 for i, so that $0 \geqslant p \cdot x_i(r) \geqslant 0$. This implies that $z_i = \langle p, \Omega_i - x_i\rangle \geqslant 0$. Since Γ is non-empty, upper semicontinuous, and convex valued, and it satisfies the appropriate boundary conditions, we may use Kakutani's fixed point theorem: Γ has a fixed point in Δ, which is a zero of ϕ. This completes the proof of existence of a quasi-equilibrium.

When all initial endowments Ω_i are strictly positive, the value of Ω_i at the equilibrium prices is also strictly positive, i.e., $\int_R p(t)\Omega_i(t)\, d\mu(t) > 0$. More generally, in a Hilbert space $p \geqslant 0$, $\Omega \gg 0$ and $\langle p, \Omega\rangle = 0$ imply $p=0$. It follows that when all initial endowments Ω_i are strictly positive, then $\langle p, \Omega_i\rangle > 0$ for all i and therefore the quasi-equilibrium is a competitive equilibrium. In particular, the value of the society's initial endowment Ω is always positive, i.e., $\langle p, \Omega\rangle > 0$. That the competitive equilibrium is Pareto efficient follows now from standard arguments (see, for example Debreu [18]). ∎

Remark. In other infinite dimensional spaces such as L_∞ or $C(R)$, the fact that all initial endowments are strictly positive does not imply the existence of a competitive equilibrium. This is because, in contrast to Hilbert spaces, in such spaces $p \geqslant 0$, $\Omega \gg 0$, and $\langle p, \Omega\rangle = 0$ does not imply $p = 0$. For example, consider the strictly positive vector $\Omega \in l_\infty$ defined by $\Omega_i = (1/2)^i$, and the positive continuous linear function defined by $p(y) = \lim_i y_i$ if y has a limit and otherwise extended to the whole space by the Hahn–Banach theorem. Then $\langle p, \Omega\rangle = 0$ even though $p \gg 0$ and $\Omega \gg 0$.

6. Existence and Pareto Optimality of Equilibrium with General Convex Consumption Sets

We turn now to the case where each agents' consumption set is a given set $K \subset H$ satisfying:

$$K \text{ is closed and convex, } K \neq H, \text{ and } y \geqslant x \in K \text{ implies } y \in K. \qquad (*)$$

A particular case is $K = H^+$, the positive orthant of H, so that K may have an empty interior. Each individual i has a utility function W_i defined on a neighborhood of K which is continuous with respect to the norm of H so that Lemma 1, which depends on continuity, and the characterization of continuous functions provided in Lemma 2, both hold. A *feasible allocation* is now a vector $x = (x_1, ..., x_k) \in K^k$ satisfying $\sum_i x_i \leqslant \Omega$. A utility vector $(W_1(x_1), ..., W_k(x_k))$ is *weakly efficient* if there is no other feasible allocation $(z_1, ..., z_k)$ such that $W_i(z_i) \geqslant W_i(x_i)$ for all i, and $W_i(z_i) > W_i(x_i)$ for some i. A *weakly efficient allocation* is one whose utility vector is weakly efficient. Note that the proof of Lemma 3 is not valid when K has no interior, because the proof requires that V, the sum of the preferred sets, has a non-empty interior. The relevant set is now the sum of the preferred sets intersected with the consumption set K, which may have an empty interior. However, the proof of Lemma 3 is still true with a modification. The following result establishes the analog to Lemma 3 for consumption sets satisfying $(*)$:

LEMMA 6. *Let $\xi = (x_1, ..., x_k)$ be a weakly efficient allocation in K^k and for $1 \leqslant i \leqslant k$, let W_i be a strictly concave and strictly increasing continuous function. Then there exists a price $p \in H^+$, $\|p\| = 1$, such that $\langle p, y \rangle \geqslant \langle p, x_i \rangle$ for all $y \in K$ satisfying $W_i(y) \geqslant W_i(x_i)$.*

Proof. Consider the weakly efficient allocation $\xi = (x_1, ..., x_k)$ in K^k and let $x = \sum_i x_i$. Let $V_i = (W_i^{x_i} \cap K)$. We must show that there exists a $p \in H^+$ with $\|p\| = 1$, supporting $\sum_i V_i$ at x. For all i the function W_i is continuous, it attains a minimum within $W_i^{x_i}$ at x_i, and is increasing so that $y < x_i \rightarrow y \notin W_i^{x_i}$. By Chichilnisky and Kalman [12, Theorem 2.1], this implies that there exists a non-zero $q_i \in H^+$ supporting the set V_i at x_i. Equivalently, there exists a non-zero $p_i \in H^+$ supporting the set $V_i - x_i$ at $\{0\}$. Consider the set $B_i = \{u \in H : \forall z \in (V_i - x_i)\ \langle u, z \rangle \geqslant 0\}$, namely B_i is the set of supports to $V_i - x_i$. B_i is convex and closed. Since K satisfies $(*)$ and W_i is increasing, $B_i \subset H^+$. If $\exists v^* \neq 0$, $v^* \in \bigcap_i B_i$, then v^* is the desired support for the set $\sum_i V_i$ at x, and thus provides the desired support for the weakly efficient allocation z.

We shall prove that $\bigcap_i B_i \neq \phi$ by induction on k, the number of agents. We saw that this is true for $k = 1$. Assume that the Lemma is true for $k - 1$ agents, and assume, to the contrary, that $\bigcap_i B_i = \{0\}$, $i = 1, ..., k$. For any given i, define $D_i = \{z \in H : \langle z, y \rangle \geqslant 0\ \forall y \in \sum_{j \neq i} V_j - x_j\}$. This is the closed convex cone of supports of the set $\sum_{j \neq i} V_j - x_j$. By the induction hypothesis $D_i \neq \{0\}$. By condition $(*)$ on K and the increasingness of W_j, $D_i \subset H^+$. Note $\bigcap_i B_i = \{0\}$ implies $B_i \cap D_i = \{0\}$ for all i, which we now assume.

Take $w \neq 0$ in D_i. Then $w + D_i \subset D_i$, so that $\forall z \in B_i$ and $y \in D_i$, $w + y \neq z$, or equivalently $w \neq z - y$. Since this is true for all $z \in B_i$ and $y \in D_i$, then w is not in the cone $B_i - D_i$. Since the set $B_i - D_i$ is closed, this implies $d(w, B_i - D_i) > 0$; by Chichilnisky and Kalman [12, Theorem 2.1], there is therefore a $p \in H$ s.t. p supports $B_i - D_i$ at $\{0\}$. Equivalently, p separates B_i from D_i, i.e., $\forall x \in B_i\ x \neq 0$, $\langle p, x \rangle \geqslant 0$, and $\forall y \in D_i\ y \neq 0$, $\langle p, y \rangle \leqslant 0$. For any such x, y, $d(x, y) > 0$, so that the separation can be made strict:

$$\langle p, x \rangle > 0 \quad \forall x \neq 0 \text{ in } B_i, \qquad \text{and} \qquad \langle p, y \rangle < 0 \quad \forall y \neq 0 \text{ in } D_i, \tag{1}$$

Since B_i is the set of all supports of the set $V_i - x_i$, (1) implies $\exists \lambda > 0$, such that $\lambda p \in V_i - x_i$. For any $\alpha < \lambda$, since $0 \in V_i - x_i$ and $V_i - x_i$ is convex, αp is in $V_i - x_i$ as well. By strict concavity of W_i, $W_i(\alpha p + x_i) > W_i(x_i)$. Similarly, (1) implies that there exists a $\gamma > 0$, such that $-\gamma p \in \sum_{j \neq i} V_j - x_j$. Let $\beta = \min(\alpha, \gamma)$. Then

$$\beta p \in V_i - x_i, \qquad -\beta p \in \sum_{j \neq i} V_j - x_j, \qquad W_i(\beta p + x_i) > W_i(x_i). \tag{2}$$

Consider now an allocation that assigns $\beta p + x_i$ to the ith agent and $-\beta p + \sum_{j \neq i} x_j$ to the rest; it is feasible because $\beta p + x_i - \beta p + \sum_{j \neq i} x_j = \sum_i x_i$. By (2) such an allocation exists which strictly increases the utility of the ith agent without decreasing that of the others, contradicting the weak efficiency of ξ. The contradiction arises from the assumption that $\bigcap_i B_i = \{0\}$. Therefore, there exists a non-zero vector v^* in $\bigcap_i B_i$. Since K satisfies (*) and W_i is increasing, $v^* \in H^+$, and provides the desired support for the weakly efficient allocation ξ, as we wished to prove. ∎

We now study the Pareto frontier when consumption sets K satisfy (*). Note that when the consumption set K is bounded below, Lemma 4 applies and there is no need for assumption (C). When K is not bounded below, however, Condition (C) is needed:

LEMMA 7. *Assume that society's endowment Ω is desirable, the consumption set K satisfies (*) and the utilities of the agents are continuous, strictly increasing, and satisfy Condition* (C). *Then on any ray r of the positive cone in R^{k+} there exists a non-zero weakly efficient utility vector. The Pareto frontier is closed and the map $v(r) = \sup_j(W^j)\ \forall W^j \in r$ is continuous in r.*

Proof. This follows directly from the proof of Lemma 5 by noting that the set $S' = \{(x_1, ..., x_k) \in F \cap K : \forall i,\ W_i(x_i) \geqslant r_k\}$ is bounded, closed, and convex under the assumptions. ∎

We now require the analog of Condition (R) in Theorem 1 for economies with a consumption set K satisfying (*). The following Condition (T) is a uniform version of the cone condition of Chichilnisky and Kalman [12], Chichilnisky [14]. Condition (T) is used to ensure that there exists a non-empty convex closed set $P'(r)$ consisting of supporting prices for each weakly efficient allocation on a ray $r \in R^{k+}$, and that the union of these sets $\bigcup_r P'(r)$ is contained in a weakly compact set which excludes $\{0\}$. This is precisely what is needed to prove the existence of a competitive equilibrium supported by non-zero prices using Negishi's arguments.[16] This condition bounds the rate of substitution between commodities [14]. It is satisfied for example by all preferences in [27].

(T) There exists a vector w, $\|w\| = 1$, which is at a distance d from the cone $C(W_i^x \cap K, x)$, for all $x \in K$ and i, $1 \leqslant i \leqslant k$, $1 > d \geqslant \varepsilon > 0$.

The equilibrium of an economy with a general consumption set K is defined as follows. A feasible allocation is a quasi-equilibrium when there is a price $p \neq 0$ with $\langle p, z - x_i \rangle \geqslant 0$ for any $z \in K$ with $W_i(z) \geqslant W_i(x_i)$, $i = 1, ..., k$. A feasible allocation is an equilibrium when it is a quasi-equilibrium and $W_i(z) > W_i(x_i) \rightarrow \langle p, z - x_i \rangle > 0$. The latter holds at a quasi-equilibrium such that $\langle p, \Omega_i \rangle > 0$ for all i. In the following theorem, Condition (C) is unnecessary when the consumption set K is bounded below, i.e., $\exists z \in H$ s.t. $\forall h$ in K, $h \geqslant z$.

THEOREM 2. *Consider an economy where each agent has a consumption set K satisfying* (*), *a desirable initial endowment $\Omega \in H^{k+}$, and such that the individual utilities $W_i: H \rightarrow R$ are continuous, strictly increasing, strictly concave, and satisfy Conditions* (C) *and* (T). *Then there exists a competitive quasi-equilibrium allocation $(x_1^*, ..., x_k^*)$ with a supporting price $p \in H^+$, $\|p\| = 1$. This allocation is a competitive equilibrium when all initial endowments Ω_i are strictly positive. The competitive equilibrium is Pareto efficient.*

[16] Condition (T) is a special case of (R), as shown in the proof of Theorem 2. Note that neither (R) nor (T) assumes that the set of *all* supports to all weakly efficient allocations is weakly bounded away from zero, but rather that there exists *some* convex set of supports for each weakly efficient allocation (within the consumption set) such that the union of these convex sets over all such allocations is weakly bounded away from zero. The reason is that if we take the positive cone H^+ as a consumption set (it satisfies (*)) then H^+ has "too many" supports, and thus the vector 0 is in the weak limit of the union of the convex sets of supports. To see this consider $H = l_2$. Then the functions (e_i), $i = 1, ...,$ defined by $e_j^i = 1$ if $i = j$, and $e_j^i = 0$ otherwise, all support H^+ and 0 is in their weak limit. We owe this remark to a referee. Note that this is not a problem of lack of existence of supports: it is, rather, a problem arising from having too many supports. The natural solution to this problem is therefore to eliminate judiciously some of the supports, and this is achieved by constructing for each weakly efficient allocation a convex set of supports in such a way that their union over all such allocations is weakly bounded away from zero. This is precisely what Conditions (R) and (T) do.

Proof. The proof follows that of Theorem 1 except that $P(r)$ in Theorem 1 is replaced by $P'(r) = \{p \in H^+ : \|p\| \leqslant 1$, p supports $x(r)$ and $\langle p, -w \rangle \geqslant \varepsilon^2$ for a given w with $\|w\| = 1\}$. We shall show that $\forall r$, $P'(r)$ is not empty. By Lemma 6 for each $r \in R^{k+}$ there exists a supporting hyperplane $p \in H^+$ for the weakly efficient allocation $x(r)$. We shall show that such a p can be chosen so that $\langle p, -w \rangle \geqslant \varepsilon^2$. As the distance between w and $C(W_i^{x_i} \cap K) - x_i$ is $d \geqslant \varepsilon$, and $\|w\| = 1$, there exists $y \in B_i =$ closure of $C(W_i^{x_i} \cap K) - x_i$, such that $\|y - w\| = d$. Convexity of $C(W_i^{x_i} \cap K) - x_i$ implies that $\langle y - w, y \rangle = 0$ and $\langle y - w, z \rangle \geqslant 0$ for all $z \in C(W_i^{x_i} \cap K) - x_i$. Let $p = (y - w)$. Then $\|p\|^2 = \|y - w\|^2 = \langle y - w, y - w \rangle = \langle y - w, y \rangle + \langle y - w, -w \rangle = \langle y - w, -w \rangle = \langle p, -w \rangle$ since $\langle p, y \rangle = 0$. Therefore $\langle p, -w \rangle = \|y - w_j\|^2 = d^2 \geqslant \varepsilon^2$ and thus $\forall r$, the set $P'(r)$ is not empty. $P'(r)$ is also convex and closed. Now consider as in Theorem 1 a sequence (p^m) in $P'(r)$. By construction, (p^m) is contained in a weakly compact set, the unit ball of H. Moreover, assumption (T) assures that the weak limit of (p^m) cannot be zero, because for each m, $\langle p^m, -w \rangle \geqslant \varepsilon^2$ and $w_j \neq 0$. The rest of the proof follows that of Theorem 1. ∎

APPENDIX: EXTENSION OF THE EXISTENCE RESULTS TO SOBOLEV SPACES H^s AND TO L_p SPACES

Let s and p be integers, $1 \leqslant s < \infty$, and $1 \leqslant p < \infty$. If $f\colon R \to R : f$ is C^s, let $\|f\|^s = \int (f(t)^2 + \cdots + D^s f(t)^2 \, d\mu(t)) < \infty$, where $D^s f$ is the sth derivative of f. The Sobolev space H^s is the completion of C^s under the norm $\|\cdot\|^s$.

$$L_p = \left\{ f\colon R \to R : f \text{ is measurable and } \int |f(t)^p| \, d\mu(t) < \infty \right\}.$$

For all $\infty > p \geqslant 1$, L_p is a Banach (complete, normed) linear space. When $\infty > p > 1$ it has the following duality property: the space of continuous linear real-valued functions on L_p denoted L_p^* is L_q for $1/p + 1/q = 1$. In particular, $L_p^{**} = L_p$ (Dunford and Schwartz [19]). One interesting feature of the Sobolev spaces H^s is that for all $s \geqslant 1$, H^s is a Hilbert, and in particular a self-dual, space with the standard inner product and countable, orthonormal coordinate basis. Furthermore, by Sobolev's theorem $H^s \subset C^k(R)$ if $s \geqslant 1/2 + k$, so that H^1 consists entirely of continuous functions, H^2 consists entirely of continuously differentiable functions, and $H^0 = L_2$ (see Adams [1], Nirenberg [29], Chichilnisky [7]). In all these spaces, therefore, prices (which are elements of the dual space H^{s*}) are also continuous or differentiable functions. All the results stated in the paper apply to H^s and L_p spaces with $\infty > p > 1$ provided the assumptions are made in the respective norms of these spaces. The main assumption is the continuity of the utility functions. The following results characterize continuous functions in L_p, $1<p<\infty$, in H^1 and H^2. As before all measures $\mu(t)$ are finite, i.e. $\int_R \mu(t) < \infty$.

LEMMA 8. *Let* $W(c)=\int_R u(c(t), t)\, d\mu(t)$, *with* u *satisfying the Caratheodory condition defined in Section 2. Then* W *defines a norm continuous function from* L_p *to* R $(1<p<\infty)$ *for some coordinate system of* L_p *if and only if* $|u(c(t), t)| \leqslant a(t)+b|\,c(t)|^p$, *where* $a(t) \geqslant 0$, $\int_R a(t)\, d\mu < \infty$, *and* $b>0$.

The proof is the same as given for the case of L_2 in Chichilnisky [7].

LEMMA 9. *Given a coordinate system for* H^i, $i=1,2$, *a function* $W(c)=\int_R u(c(t), t)\, d\mu(t)$ *is continuous from* H^1 *to* R *if and only if the conditions in Lemma 8 are satisfied for* u *and for* Du. W *is continuous from* H^2 *to* R *if the conditions in Lemma 8 are satisfied for* u, Du, *and* D^2u.

The proof is the same as in Chichilnisky [8].

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