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COMMUNITY PREFERENCES AND SOCIAL CHOICE*

Graciela CHICHILNISKY and Geoffrey HEAL

Columbia University, New York, NY 10027, USA

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This paper gives necessary and sufficient conditions for the aggregation of preferences, extending an earlier treatment of aggregation by Stolper, Gorman, Samuelson and Chipman. Such aggregation procedures are intended to deal with the problem of aggregating demand functions in econometrics, where the aggregate is required to be independent of the income distribution. Thus, it is usually assumed in this form of aggregation that all consumers face the same prices, but that the distribution of income is unrestricted.

In order to establish the characterisation result, we present a new approach to preference aggregation which involves summing certain subsets of the graphs of the preferences, viewed as subsets of a Euclidean space. This procedure has a clear geometrical interpretation, and a number of useful applications. In particular, it enables us to analyse the possibility of aggregation when prices are not constrained to be the same for all consumers, a case of possible empirical significance. We also show that the Stolper–Gorman–Samuelson–Chipman construction of community indifference curves coincides with a special case of this procedure.

Finally, this approach allows us to develop the relationship between these forms of aggregation and the preference aggregation problem as it occurs in social choice theory.

1. Introduction

This paper has three aims. One is to provide necessary and sufficient conditions for the aggregation of preferences, extending earlier approaches by Stolper (1950), Gorman (1953), Samuelson (1956) and Chipman (1974) to the analysis of social indifference curves. Such aggregation procedures are intended *inter alia* to deal with the problem of aggregating demand functions in econometric analysis, when the aggregate is required to be independent of the distribution of income, and all consumers are assumed to face the same prices. This is the subject of Theorems 3 to 5 below.

Our second aim is to analyse the possibility of aggregating preferences when different individuals face different prices (the subject of Theorem 2). This is arguably an important empirical case, given differences in tax rates, the existence of concessionary prices, price discrimination, etc. There are in fact two sub-cases. One is where we know *a priori* that individual i will face a fraction α_i of the general price level (pensioners pay half price; the rich get

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a bigger tax deduction). The other is where we can place no restrictions at all on the distribution of prices across individuals. Our analysis could be used to provide a characterisation of both cases, though in fact we only deal explicitly with the latter, as this is the more demanding case.

Our third aim is to study the relationship between these forms of preference aggregation and the aggregation problem of social choice theory. We show that the former is a particular case of a recent version of the latter, in that the necessary and sufficient conditions for the existence of social indifference curves which give a well-defined social preference, are a special case of conditions given in Chichilnisky (1980) and Chichilnisky and Heal (1979) for the existence of social choice rules which are continuous, anonymous and which respect unanimity.

The plan of the paper is as follows. In section 2 we define a method of aggregating preferences by summing their graphs. This is conceptually an easy and obvious way of aggregating, and we present conditions under which such aggregation is well-defined, i.e., under which the sum of the graphs of preferences, is the graph of a preference.[1] This mode of aggregation does not coincide with that of Stolper, Gorman, Samuelson and Chipman, but corresponds to aggregation when there is no restriction on the distribution of prices across individuals.

In section 3 we formalize the Stolper–Gorman Samuelson (SGS) approach in terms of operations on the graphs of the preferences, and show that this involves adding certain subsets of the graphs, and then taking the union of the resulting sets. By using graph aggregation, it turns out to be possible to give a very simple geometrical characterization of the SGS approach, and also of the conditions under which it yields a well-defined preference. In fact we are able to extend earlier analyses by giving conditions for preference aggregation which are necessary as well as sufficient. We are also able to characterise the conditions under which an aggregate preference will be complete, an issue that has not previously been discussed.

In section 4 we relate the rules for aggregating preferences to operations on vector fields. This enables us to relate the results of the previous sections to the literature on social choice theory. Some of the more technical results are contained in an appendix.[2]

[1]For a discussion of the representation of preferences by their graphs, see Chichilnisky (1977).

[2]It is worth noting that our concern here is with the aggregation of families of indifference surfaces. In other words, we are concerned with the aggregation of families of level sets of real—valued functions. Although it is conventional in this literature to think of these as being generated by utility functions, they could equally well be generated by real-valued (i.e., single output) production functions: all of the results established below on the aggregation of preferences could therefore be applied equally well to the aggregation of production functions. Recent work by Hildenbrand (1979) studies aggregation of production functions as a problem in set addition.

2. Graph aggregation of preferences

It is shown below that the form of preference aggregation considered by Stolper, Gorman, Samuelson and Chipman may be formalised as follows. Let x be a point in the choice space. The set of points preferred to x according to the social or aggregate preference is obtained as follows. Take the vector sum of the sets of points preferred by each individual i to certain choices x_i, where the x_i summed over i equal x. Repeat this procedure for all possible sets of choices x_i that add up to x, provided that at these choices x_i all consumers have the same marginal rates of substitution. Then take the union of these sums over all sets of x_i. In many contexts the equality of MRSs is a natural restriction, and corresponds to the assumption that the only choices of interest are those where all consumers face the same prices.

Formally, if $P(x)$ denotes the set of points preferred to x according to the aggregate preference, and $P_i(x_i)$ is that preferred to x_i by individual i, then

$$P(x)=\bigcup_{\{x_i:\sum_i x_i=x\}}\left\{\sum_i P_i(x_i)\right\}:T_i(x_i)=T_j(x_j),$$

where the sums are over individuals, and where $T_i(x_i)$ denotes the normal[3] to the indifference surface of the ith preferences at x_i.

This procedure is a special case of a more general aggregation method, where the aggregate preferred set $P(x)$ is generated by summing the preferred sets $P_i(x_i)$ of individuals for all combinations of x_i which sum to x, without restriction on marginal rates of substitution. Formally, this latter procedure gives

$$P(x)=\bigcup_{\{x_i:\sum_i x_i=x\}}\left\{\sum_i P_i(x_i)\right\},$$

where the sum is over individuals. This definition is of interest for two reasons. One is that it has a very straightforward technical interpretation: it is shown in Lemma 3 below that it corresponds precisely to summing the graphs of the preferences in Euclidean space, a very natural form of aggregation.

It also has a straightforward economic interpretation. Constructing the aggregate preferred set at x by summing those of individuals at all combinations of points x_i which sum to x, independent of marginal rates of substitution, corresponds to the construction of an aggregate preference when there is no restriction at all on the distribution of prices over individuals. *Thus if the sum of the graphs of the individual preferences is the graph of a preference, then we can find an aggregate preference and an aggregate*

[3]As noted formally in the next section, this approach requires a smoothness condition on preferences.

demand which are independent of the distribution of income for any distribution of prices across people. (The aggregate may of course depend on the distribution of prices: but for any distribution of prices it will not depend on the distribution of income.) As we shall see, the conditions for this are naturally more restrictive than those for aggregation when price vectors are known to be the same for everybody.

In the remainder of this section, we explore the properties of preference aggregation via the summation of graphs. A *preference* on R_n^+, the positive cone of R_n, may be identified with a subset of $R_n^+ \times R_n^+$, known as the *graph* of the preference: if xp_iy means that agent i prefers x to y, then the graph of this preference, denoted by G_i, is defined as

$$G_i = \{(x, y) \in R_{2n}^+ : xp_i y\}.$$

We discuss summation of graphs in the usual sense of vector summation, so that if X and Y are subsets of R_{2n}^+, then

$$X + Y = \{z \in R_{2n}^+ : z = x + y,\ x \in X,\ y \in Y\}.$$

If we have a finite collection of preferences represented by subsets of R_{2n}^+, it is clear that these subsets can always be added and a sum defined. A natural question, then, is: Under what circumstances is the aggregate object constructed in this way, the graph of a preference? Since a subset of R_{2n}^+ can always be interpreted as a relation on R_n^+, the question is, more precisely: Under what circumstances does the aggregate relation have the properties of a preference ordering? In the following, we require the aggregate relation to be transitive and irreflexive, and show that the sum of the graphs is always a transitive relation, but need not be an irreflexive one. We then derive conditions which in our context are necessary for the aggregate relation to be complete, transitive and irreflexive. These conditions are very restrictive: they are that individual preferences are identical and homothetic. In the next section, we consider a slightly different approach to summing the graphs of preferences, a generalization of the construction in this section. The approach of section 3 in fact yields an aggregate preference under rather weaker conditions than does the approach of this section.

In the following subsection we introduce notation and definitions, state the assumptions we shall be making on preferences, and explore their implication.

2.1. Definitions and assumptions

We shall consider preference orderings on R_n^+, the positive cone of a commodity space R_n. There is a group of M agents, indexed by i, $i = 1, 2, \ldots, M$, and, as before, if the ith agent prefers x to y we write xp_iy. If the agent is indifferent, we write $xi_i y$, and we denote preference or indifference by $xpi_i y$. $P_i(x)$ stands for the set of points in R_n^+ strictly preferred to x, that is,

$$P_i(x) = \{y : yp_i x\}.$$

The preferences that we shall consider will always satisfy the following conditions:

(A.1) *Continuity*: The sets $\{x \in R_n^+ : xpi_i y\}$ and $\{x \in R_n^+ : ypi_i x\}$ are closed for any $y \in R_n^+$.

(A.2) *Thin indifference curves*: $\{x \in R_n^+ : xi_i y\}$ has measure zero, for any y and each i.[4]

(A.3) *Transitivity of the strict preference*: For any $x, y, z \in R_n^+$, $xp_i y$ and $yp_i z$ imply $xp_i z$.

(A.4) *Irreflexivity of the strict preference*: For any $x \in R_n^+$, it is not the case that $xp_i x$.

The above four assumptions will be maintained at all points of this paper, and will be the only assumptions used for much of section 2. We shall, however, have occasion to use the following later in this section:

(A.5) *Completeness*: For all $x, y \in R_n^+$, either $xp_i y$ or $yp_i x$ or $xi_i y$.

(A.6) *Homotheticity*: There exists $x_0 \in R_n^+$ such that for all $x \in R_n^+$, $i_i(x) = \{x' : x' i_i x\}$ is given by $i_i(x) = \lambda i_i(x_0)$, for some scalar $\lambda > 0$.

Under homotheticity, indifference curves are thus radial projections: any real-valued representation of the preference will be by a homogeneous function.

The next step is to investigate the implications of these assumptions for the structure of the graph G_i of a preference that satisfies them. We denote by RS the reflexive space in R_{2n}, the diagonal of $R_n \times R_n$, i.e., $RS = \{x, y : x = y\}$. The following are immediate implications of the various assumptions on preferences, (A.4) and (A.3):

$$\text{Irreflexivity} \Rightarrow G_i \cap RS = \phi.$$

$$\text{Transitivity} \Rightarrow (b, a) \in G_i,\ (c, b) \in G_i \Rightarrow (c, a) \in G_i.$$

[4]This is a slightly stronger version of the usual assumption of non-thick indifference curves.

2.2. Conditions for aggregation

We shall now investigate a method of aggregating preferences by summing their graphs. Thus, given a set of preferences, we take the graphs G_i, $i=1,2,\ldots,M$, and construct the sum

$$G=\sum_{i=1}^{M} G_i.$$

This is a subset of R_{2n}, and can therefore be interpreted as the strong part of a relation on R_n^+.

The question we have to analyze is then: Under what conditions on G_i, $i=1,2,\ldots,M$, is G a preference? The following lemma establishes that transitivity is preserved by the summation of graphs. Its proof is immediate.

Lemma 1. If G_i, $i=1,2,\ldots,M$, are transitive, then so is G.

From this we know that G will not fail to be a preference because of intransitivity. However, we have to check irreflexivity: if G intersects RS, there will be points strictly preferred to themselves and hence in the interiors of their preferred sets.

Theorem 1. Let the G_i satisfy (A.1) to (A.4). A sufficient condition for the sum of their graphs to define an aggregate preference, is that the G_i are all contained in an open halfspace in R_{2n} which is supported by the origin and has RS in its boundary.

Proof. We have to show that the aggregate relation G will be transitive and irreflexive. Transitivity follows from Lemma 1. To prove irreflexitivity, we have to demonstrate that the intersection of G with RS is empty. Let S be the open half-space in R_{2n} meeting the conditions of the theorem. Then, by assumption,

$$S\cap RS=\phi \quad \text{and} \quad G_i\subset S, \quad \text{for all } i.$$

But an open half-space is closed under the operation of vector addition. Hence,

$$G=\sum_i G_i\subset S \quad \text{and so} \quad G\cap RS=\phi,$$

which completes the proof.

Lemma 2. If preferences are complete and have thin indifference curves [i.e., satisfy (A.2) and (A.5)], then for almost every $(x,y)\in R_{2n}^+$, either $(x,y)\in G_i$ or $(y,x)\in G_i$.

Proof. This follows immediately from the measure zero form of Fubini's Theorem, as given in, for example, Guillemin and Pollak (1973, p. 204).

The next theorem gives an economic interpretation to the conditions of Theorem 1. Note that Theorem 1 gives conditions sufficient for aggregation *even when there is no restriction on the distribution of prices across individuals* (see the comments after Lemma 3).

Theorem 2. If preferences satisfy the sufficient conditions of Theorem 1 and are complete, then they are homothetic and identical almost everywhere.

The proof of this theorem is given in the appendix. Being homothetic and identical almost everywhere is a necessary condition for the satisfaction of a sufficient condition (that given in Theorem 1). It is itself therefore neither necessary nor sufficient. However, the conditions of Theorem 1 are extremely close to being necessary for aggregation (they are necessary up to certain minor but complex technicalities), so that it is reasonable to regard being homothetic and identical almost everywhere as necessary. It should perhaps be emphasized that the argument so far has made no use of convexity or differentiability assumptions.

The conditions of Theorem 2 are exactly those given by Samuelson (1956), but they solve a more demanding problem, namely that of aggregating preferences when individuals do not all face the same prices. The following section analyzes a formalization of the alternative procedure.

The construction of an aggregate preference $G=\sum G_i$ is a way of assigning to each point $x \in R_n^+$ a preferred set $P(x)$. The next lemma characterizes the way in which this set is constructed. The proof is immediate.

Lemma 3. Let $P(x)$ be the preferred set for a point x according to the preference $G=\sum G_i$. Then

$$P(x)=\bigcup_{\{x_i,\ \sum x_i = x\}} \left\{\sum_{i=1}^{M} P_i(x_i)\right\}.$$

Under the procedure studied, then, the preferred set to a choice x is constructed by summing the sets which, according to the individual preferences, are preferred to M points which sum to x, repeating this for all sets of M points summing to x, and then taking the union of the resulting sets.

The Stolper–Gorman–Samuelson approach is, as mentioned, different. We show below that in this case, the union is taken not over *all* possible combinations of points that sum to x, but *only* over those points which sum to x and *at which the various individuals have the same marginal rate of substitution* [Samuelson (1956, p. 8)], i.e., for preferences with a well-defined normal to the indifference surface at each choice,

$$P(x)=\bigcup_{\substack{\{x_i:\sum x_i=x\} \\ T_i(x_i)=T_j(x_j)}}\left\{\sum_i P_i(x_i)\right\}.$$

Here $T_i(x_i)$ is the normal to i's indifference surface at x_i. Obviously, since in this latter case the union is taken over a smaller set of points, the loss of reflexivity is less likely to occur. We shall see in the next section that in fact this gives a little extra scope for successful aggregation. As noted, this more restrictive approach corresponds to an assumption that all consumers face the same prices, an assumption *not* implied by the construction of Theorems 1 and 2, where there is no requirement that the points whose preferred sets we summed, should have equal tangents. Theorems 1 and 2 tackle a more general problem.

3. Social indifference curves

In this section we explore the properties of a rather different method of aggregating the graphs of preferences. Instead of summing the entire graphs, as in the previous section, we sum only subsets of these. This procedure is a formalization of the Stolper–Gorman–Samuelson approach to the construction of social indifference curves. As mentioned above, this amounts to a rather different way of constructing the preferred set associated with any point x. We now limit our attention only to those sets of points that sum to x *and* are characterized by equal tangents for all individuals. This means, of course, that we must restrict the analysis to smooth preferences (a restriction not needed before).

The procedure adopted here can be described as follows. We partition the graph of a preference into subsets, each corresponding to the preferred sets associated with points with a given most-preferred direction or gradient. We then sum over all individuals those subsets characterized by a given gradient and repeat for all gradients. Next, we take the union of the sums over all gradients: this gives the graph of the aggregate preference.

This approach always leads to a relation on R_n^+, but this need not be transitive, irreflexive or complete. Our concern is therefore to specify conditions under which these properties hold. In this case irreflexivity always holds and is relatively easy to establish, whereas transitivity poses problems. However, both can be shown to hold under conditions weaker than those of the last section. In particular, a sufficient condition for the aggregate relation to be a preference is that the individual preferences are what we term *affinely homothetic and identical*. This concept is defined formally below, and means essentially that all preferences are affine translations of a given homothetic preference. This condition is also a necessary condition under the usual assumption that the expansion paths of an individual's preference all have a unique point of intersection. This will be discussed below. In addition to assumptions (A.1) to (A.4), two further assumptions on preferences are maintained throughout this section:

(A.7) *Strict convexity*: $\forall x$, $\{y: y p_i x\}$ is a strictly convex set.

(A.8) *Regularity*: A preference is representable by a C^2 regular utility function whose first derivative is regular on the domain of definition.[5]

Denote by $G_i(x^0)$ the set of pairs (y, x^0) in G_i, the preferred set to x^0. The closure of this contains (x^0, x^0). Let $T_i(x^0)$ be the unit normal to individual i's indifference surface at x^0. G_i/T will denote the subset of the graph G_i consisting of the union of all preferred sets to all points where the normal to an indifference surface has a fixed value T, i.e.,

$$G_i/T = \bigcup_{\{x: T_i(x) = T\}} G_i(x).$$

An aggregate preference is now defined as a certain subset of $G = \sum_i G_i$,

$$G^* = \bigcup_T \left\{ \sum_{i=1}^{M} G_i/T \right\}.$$

This set G^* is obtained by summing the preferred subsets for all points with a given normal and taking the union for all normals.[6]

[5] A function is regular if at each point of its domain, its derivative (a linear map) is onto.

[6] This formalizes the aggregation procedure in Gorman (1953) and Samuelson (1956). These authors construct the preferred set to a point X by summing the individual preferred sets to points x_i which sum to x and are all chosen at the same prices. This is clearly the same operation as that in the definition of G^*. We mentioned in the previous section that the usual approach to aggregation can also be formalised by constructing the preferred sets to X according to

$$P(x) = \bigcup_{\substack{x_i \sum x_i = x \\ T_i(x_i) = T_j(x_j), \forall i,j}} P_i(x_i)$$

In fact it is clear immediately that if this approach is to yield a single tangent to the aggregate indifference surface at x, then there must be only one tangent which is common to m-tuples of point x_i summing to x. In this case one is again constructing the aggregate preferred-to-x set by summing individual preferred-to-x_i sets where the x_i sum to x and are all chosen at a given price vector.

Lemma 4. *The aggregate relation G^* is irreflexive.*

Proof. By assumption the individual preferences are irreflexive. Hence,

$$(G_i/T) \cap RS = \emptyset.$$

Now,

$$\left(\sum_i G_i/T\right)(x^0) = \left\{(x, x^0) : (x, x^0) \in \sum_i G_i/T\right\}$$

$$= \left\{(x, x^0) : (x, x^0) = \sum_i (x_i, x_i^0),\right.$$

$$\left.(x_i, x_i^0) \in G_i/T,\ \sum_i x_i^0 = x^0\right\}$$

$$= \left\{(x, x^0) \in \bigcup_{\{x_i^0\}: \sum_i x_i^0 = x^0} \sum_i (G_i/T)(x_i^0)\right\}.$$

where

$$(G_i/T)(x_i^0) = \{(y, x_i^0) \in G_i/T\}.$$

So we may write

$$\left(\sum_i G_i/T\right)(x^0) = \bigcup_{\{x_i^0\}: \sum_i x_i^0 = x^0} \sum_i (G_i/T)(x_i^0). \tag{A}$$

Now consider a typical element of the right-hand side,

$$\sum_i (G_i/T)(x_i^0).$$

Let $\bar{X}$ be the *closure* of a set X. The normal to the closure $\overline{(G_i/T)}(x_i^0)$ of $(G_i/T)(x_i^0)$ at (x_i^0, x_i^0) is T. To see this, note that

$$(G_i/T)(X_i^0) = \{(x_i, x_i^0) \in G_i/T\} = G_i(x_i^0),$$

where $T_i(x_i^0) = T$. This is a set contained in an affine subspace A_i of dimension n, and can be identified by projection onto its first n components with the preferred set $P_i(x_i^0)$ to x_i^0 under the ith preference. Hence, the normal of the indifference surface at x_i^0, which by construction is T, is a normal to the support of $G_i(x_i^0)$ at (x_i^0, x_i^0) within A_i.

It now follows by convexity that the sum of the closures $\overline{(G_i/T)}(x_i^0)$,

$$\sum_i \overline{(G_i/T)}(x_i^0),$$

has a support in $A=\sum A_i$ with normal T at the point (x^0, x^0), where $x^0=\sum_i x_i^0$.[7]

This is true for every set in the union taken on the right-hand side of (A), so that

$$\sum_i \overline{(G_i/T)}(x_i^0)$$

has T as normal to a support at (x^0, x^0). Hence (x^0, x^0) is not in the open set $(\sum_i G_i/T)(x^0)$. Since any other element of RS is of the form (x', x') with $x' \neq x^0$, this set has an empty intersection with RS, that is,

$$\left(\sum_i G_i/T\right)(x^0) \cap RS = \phi.$$

But

$$(G^*/T)(x^0) = \left(\sum_i G_i/T\right)(x^0),$$

so

$$(G^*/T)(x^0) \cap RS = \phi.$$

As this is true for all x^0, the result follows.

We next look at the points in the closure of G_i/T that lie in the reflexive space RS: formally, we look at

$$\overline{(G_i/T)} \cap RS.$$

This is the set of points (x, x) such that the preferred set to x, $G_i(x)$, has a given normal T at x. It is clearly isomorphic to the set of points in the commodity space where the individual indifference curves have a normal T. This is what Gorman (1953) calls an *Engel curve*, and what we shall term a *tangency path*, as it shows tangencies between indifference surfaces and budget lines at various income levels and at constant relative prices given by T.

We use the term tangency path, rather than the more familiar terms Engle curve or expansion path, quite deliberately, as the two may differ. An expansion path gives the locus of consumption bundles chosen by a utility-maximising consumer at fixed prices as his or her income level varies. If non-negativity constraints on the consumption choice are binding, so that the chosen bundle is on the boundary of the positive orthant, then the choice

[7]In fact what we are using here, is that the sum of boundary points with a common tangent, is a boundary point in the sum.

will not, in general, be at a point of tangency between a budget line and an indifference curve. The expansion paths will thus contain segments in the boundaries of the positive orthant not characterised by tangencies: fig. 1 illustrates this. Though these boundary segments may be relevant for demand aggregation, they are not so for preference aggregation, as the aggregate preference is constructed by assigning to a point $x \in R^n$ a preferred set given by the sum of the preferred sets to points x_i which sum to x *and in addition* are characterised by a specific tangent to the indifference surface at x_i. If x_i is a boundary point, by the smoothness of preferences we may assign to it a tangent given by the limit of tangents to indifference surfaces at a sequence of points convergent to x_i. Consequently, we define a tangency path to contain only true points of tangency between indifference surfaces and budget hyperplanes of a given slope.

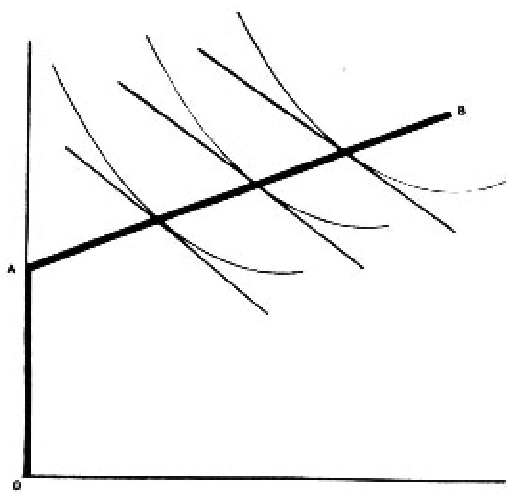

Fig. 1. OAB is an expansion path, but AB is a *tangency path*. The tangent to an indifference surface at A is the limit of the tangents at a sequence of points converging to A from inside the positive cone. All points along AB thus have the same tangent, and this is in general different from the tangent at points along OA.

For homothetic preferences, these paths are rays, as all scalar multiples of a point have the same normal. More generally, under regularity assumptions, these sets are manifolds of dimension one in R_n, which we look at imbedded in RS, the diagonal of $R^n \times R^n$. To see this, let U_i be the utility function representing individual i's preferences, and DU_i be the first derivative. Then,

$$DU_i : R_n \to R_n.$$

However, only normalized gradient vectors are considered, as we work with ordinal preferences. Thus, two points in the choice space R_n are said to have the same tangent if the normals to the indifference curves at those points are parallel, even though the normals to these may be of different magnitudes. The normalization of the gradient vectors of U_i gives a map from R_n into S^{n-1}, the sphere in R_n; all colinear vectors increasing in the same sense are now identified. Let N be the normalisation operation; then

$$N: R_n - \{0\} \to S^{n-1},$$

and a tangency path is the inverse image of a point in S^{n-1} under the composition of DU_i and N denoted $N \circ DU_i$.

By assumption (A.8), $N \circ DU_i$ is a regular C^1 function, so the inverse image of a point on S^{n-1} is a C^1 manifold in R_n [Hirsch (1975, p. 22, theorem 3.2.)]. Also, by Guillemin and Pollack (1974, p. 21), the inverse image is a manifold of dimension 1. The tangency paths are thus smooth curves in R_n^+, and therefore

$$\overline{(G_i/T)} \cap RS$$

is a smooth curve in RS, the diagonal of $R_n \times R_n$.

It is clear that if preferences are smooth and transitive, there can be only one tangency path through any point. Formally, if T' and T'' are two distinct normalized gradient vectors,

$$[\overline{(G_i/T')} \cap RS] \cap [\overline{(G_i/T'')} \cap RS] = \phi.$$

If the aggregate relation is to be transitive, we need to impose conditions which ensure that the set G^* has this property also.

The essence of this problem is that we have several families of curves in RS, given by the tangency paths of the individual preferences, one family per preference. Each family is non-intersecting, fills the space, and is indexed by T. We sum these by summing over all families the curves with a given index T, and require that the family so constructed has the non-intersecting property. Formally,

$$\overline{(G_i/T)} \cap RS$$

defines a family of curves in RS, and we look at the aggregate family whose typical member is obtained as

$$\sum_{i=1}^{M} \overline{(G_i/T)} \cap RS.$$

For transitivity of the aggregate relation, it is *necessary* that this is again a family of non-intersecting curves. In general, of course, the vector sum of the graphs of two or more curves is not the graph of a curve — e.g., the sum of two non-parallel lines in R_2 is a two-dimensional set. However, Lemma 5, given in the appendix, contains necessary and sufficient conditions for this aggregate to be a family of curves having the non-intersection property. The economic implications of Lemmas 5 and 6 are summarized in the following theorem (Theorem 3), which states these conditions in terms of the properties of consumer expansion paths. Theorem 3 represents a generalization of Theorem 3 of Chipman (1974), which gives related sufficient conditions. We are now, of course, assuming as in the earlier literature, that all consumers face the same prices.

Theorem 3. The following are necessary and sufficient conditions for the aggregate relation G^* to define a transitive and irreflexive preference. For each individual i the tangency paths on the positive orthant should be a set of the form

$$\{(t, y) \in R^n : y_i = \alpha_i(T) + \beta(T)t_i,\ i = 1, \dots, n\},$$

where

$$y_i \in R_{n-1}, \quad t_i \in R_1, \quad \beta(T) \in R_{n-1}, \quad \alpha_i(T) \in R_{n-1},$$

and the tangency path intersects in the negative cone.[8]

Proof. This follows immediately from Lemmas 5 and 6 of the appendix.

Thus the tangency paths of all preferences should be straight lines, and parallel across preferences for any given set of prices. Homothetic and identical preferences, as discussed in the previous section and stipulated by Samuelson, meet these conditions, but such strong restrictions are not in general necessary. If the preferences considered are complete, the possibility of inferior goods is ruled out by the condition that tangency paths cross only in the negative orthant. Note however that completeness is neither assumed of individual preferences, nor established for the aggregate.

A special case considered by Gorman is the following:

[8] I.e., the sets of vectors with no component positive.

(A.9) For any x^0, $\{x: U_i(x) \geqq U_i(x^0)\}$ is interior to the positive orthant. Thus no indifference curves may cut an axis.

In this case, expansion paths must pass through the origin, in which case we have:

Theorem 4. If individual preferences satisfy (A.9), then a necessary and sufficient condition for G^* to be transitive and irreflexive, is that all preferences are identical and homothetic.

Proof. Apply Theorem 3, noting that $\alpha_i(T)=0$, for all i.

This is presumably the case envisaged by Samuelson and also by Chipman (1974, theorem 3) who gives sufficient conditions for aggregation. Note that in this case if the individual preferences are complete, then it is easily shown that the aggregate ordering is complete, whereas under the less restrictive conditions of Theorem 3 this is not always true. Further conditions are then needed for completeness, and we now investigate them.

First, consider preferences that have the property that the tangency paths all intersect in a single point (outside of the positive orthant).

Lemma 7. A family of preferences with tangency paths of the form

$$y_i = \alpha_i(T) + \beta(T)t_i$$

has only a single point of intersection of these paths in the negative cone if and only if the paths are of the form

$$(y_i + A_i) = \beta(T)(t_i + B_i),$$

where $\beta(T)$ is as above, $A_i \in R^{n-1}$, $B_i \in R$, $A_i \leqq 0$ and $B_i \leqq 0$.

Proof. This is obvious from the fact that all paths must meet at one point, say, $(A_i, B_i) \in R^n$, which is then taken as the origin for writing the equations of the tangency paths. If the intersection is within the negative cone, all components of (A_i, B_i) are non-positive.

In economic terms, Lemma 7 implies that if preferences have the degree of similarity required for aggregation, and all have a single point of intersection of their tangency paths, then they are all generated by taking a given homothetic preference, translating its origin into the negative cone, and then taking as the preference the restriction of this to the non-negative orthant. In this case, we will term the preference family *affinely homothetic and identical.*[9]

Theorem 5. If preferences are such that their tangency paths have a single point of intersection, then a necessary and sufficient condition for the G^* aggregate to be complete, irreflexive and transitive, is that the preferences are affinely homothetic and identical.

Proof. For irreflexibility and transivity, this is merely a restatement of Theorem 3. Completeness is evident, as the new preference has expansion paths of the form

$$\left(y+\sum_i A_i\right)=\beta(T)\left(t+\sum_i B_i\right), \qquad y=\sum y_i, \quad t=\sum t_i,$$

which are again affinely homothetic and identical, with the point of intersection the sum of the points for the individual preferences. This establishes the theorem.

It is worth remarking at this point that there is a difference between the conditions needed for preference aggregation, and those needed for demand aggregation. Our results are of course concerned with the former. This difference can be best seen by referring to fig. 1. The theorems above show that preference aggregation requires that *tangency paths* such as AB should be parallel across individuals at a given set of prices: demand aggregation, however, requires that *expansion paths* such as OAB satisfy this property. This is clearly a stronger restriction, and accounts for the fact that the necessary and sufficient conditions given in our Theorems 3 and 5 are weaker than equivalent conditions given for demand aggregation, as for example in Theorem 4.5 of Chipman and Moore (1973). It is clear that this difference arises from the existence of boundaries to the consumption set, and that, if the consumption sets were unbounded, then the two problems would be the same.

[9]This is similar but not identical to Gorman's (1978) concept of quasi-homotheticity. Quasi-homotheticity is a property of individual preferences, and requires that they have linear Engel Curves. In general, our results appear to be closely related to those of Gorman (1953), though there are significant differences. The introduction of Gorman's paper (para. 2, p. 53) gives a heuristic statement of his results which is clearly similar to Theorem 5 above. However, the formal characterisation of preference families which can be aggregated, given in his Theorems 9 to 11, p. 73, appears to be quite different and is given by a set of differential equations. Gorman's analysis also differs from ours in not considering the issue of completeness, and in not noting the distinction between preference families which do or do not have unique points of intersection of their families of tangency paths. Of course, our methodology is also different: it yields aggregation results for the cases when individuals may face different prices, and also gives results in a form which allows a comparison with aggregation problems in social choice theory.

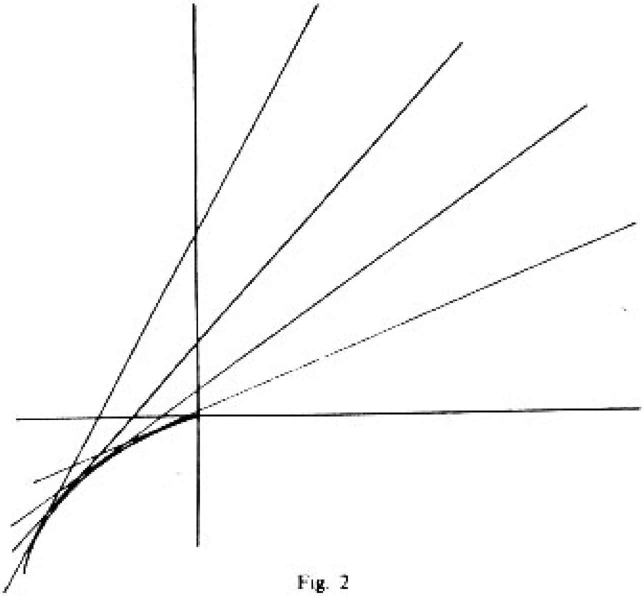

Fig. 2

It is clear that preferences need not have tangency paths with a single point of intersection, as fig. 2 shows. In this case, Theorem 3 still applies.

The condition that preferences by affinely homothetic and identical is a simple and intuitively appealing condition in geometrical terms. It can also be given a very straightforward economic interpretation. We shall show below that if preferences are affinely homothetic and identical, then they can in effect be made identical by some suitable reallocation of the individuals' initial endowments. In other words, a necessary and sufficient condition for the existence of an aggregate preference in the sense of this section is that all individual preferences can be made identical to a specified homothetic preference by a suitable reallocation of initial endowments.

To interpret the condition of being affinely homothetic and identical in more detail, we need a definition:

A family of preferences represented by a set of continuously differentiable ordinal utility functions $(u_1, u_2, \ldots, u_m)$ is said to be *affinely homothetic and identical* (*A.H.I.*) if there exist a continuously differentiable linear homogenous utility function u_0 and a set of constant vectors $Z_i \in R_n$, such that for all $x \in R_n^+$,

$$u_i(x) = u_0(x + Z_i), \qquad i = 1, 2, \ldots, m.$$

The tangency path of a member of such a family is given by the set of points where its normalized gradient vector assumes a particular value, i.e., the tangency path is of the form

$$\{x \in R_n : \|Du_i(x)\| = T\}.$$

But by definition, this equals

$$\{x \in R_n : \|Du_0(x + Z_i)\| = T\} = \{x \in R_n : x + Z_i = y, \|Du_0(y)\| = T\},$$

as u_0 is linearly homogeneous. Fig. 3 illustrates this set for the case $n = 2$: it is just an affine translation of the ray alone which $\|Du_0(y)\| = T$. For a similar use of affinely translated homogeneous functions, see Brown and Heal (1980). We can now return to the economic interpretation of this condition. Consider the following utility-maximization problems, where w_i is individual i's endowment vector:

$$\max u_0(x) \quad \text{subject to} \quad p \cdot x \leqq p \cdot w_0, \tag{Q_0}$$

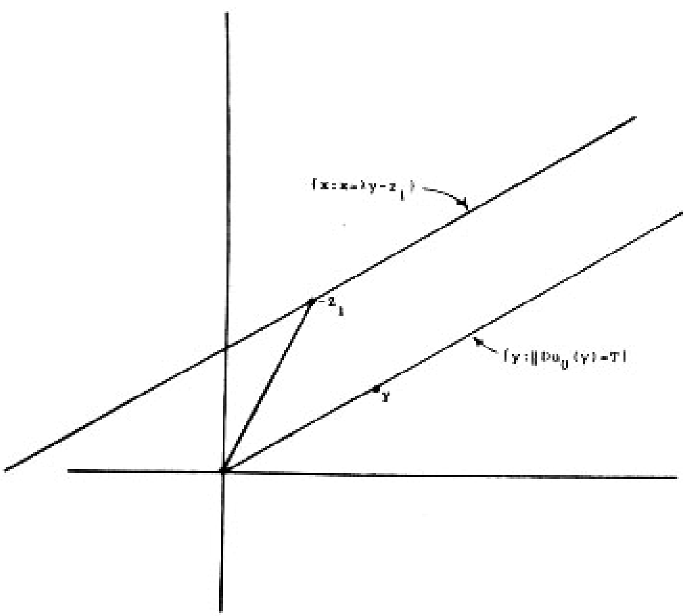

Fig. 3

$$\max u_i(x) \quad \text{subject to} \quad p\cdot x \leqq p\cdot w_i. \tag{Q_i}$$

Now, if we are considering an affinely homothetic and identical family, (Q_i) can be rewritten as

$$\max u_0(x+Z_i) \quad \text{subject to} \quad p\cdot x \leqq p\cdot w_i. \tag{Q_i}$$

Define a new variable $q_i = x + Z_i$. Then (Q_i) can be expressed as

$$\max u_0(q_i) \quad \text{subject to} \quad p\cdot(q_i - Z_i) \leqq p\cdot w_i, \tag{Q_i}$$

that is,

$$\max u_0(q_i) \quad \text{subject to} \quad p\cdot q_i \leqq p\cdot w_i + p\cdot Z_i. \tag{Q_i^*}$$

By comparison of the problems (Q_0) and (Q_i^*), we see that the budget problem of a typical member of a group of individuals with affinely homothetic and identical preferences is identical with that of the individual with homothetic preferences u_0 from which the remainder of the family were generated, subject to the amendment of individual zero's budget constraint by an amount $p\cdot Z_i$. *Thus, if preferences form an A.H.I. family, the budget problems of individuals can be reduced to a single budget problem of an individual with homothetic preferences by changes in their initial endowment vectors.* So intuitively one might say that, when tangency paths have a single point of intersection, *the necessary and sufficient condition for preferences to be aggregated is that the differences among them, and their departures from homotheticity, can be removed by suitable changes in initial endowments.*

In this section we have reviewed a method of aggregating preferences by summing certain subsets of their graphs. The subsets selected, namely those that were preferred sets to points with a given normal, had a natural economic interpretation. However, it is clear that many other forms of aggregation could be defined by selecting different subsets of the graphs. For example, in his Theorem 4 Chipman (1974) considers the aggregation of homothetic preferences when the distribution of income is fixed. This can easily be formalized as summing rather different subsets of the graphs from those that we considered. In particular, it corresponds to summing subsets of the graphs which are preferred sets to points which all have a given normal and also satisfy the income distribution rule. Chipman establishes that homotheticity is sufficient for aggregation with a fixed income distribution. It is clear from the above that homotheticity is not necessary: with a fixed distribution of income, we can aggregate preferences that form an A.H.I. family. No member of such a family need be homothetic.

4. Graph aggregation and social choice theory

We shall prove that the results on graph aggregation, and in particular those on the Gorman–Stolper–Samuelson aggregation rule, can be related to recent results in social choice theory given in Chichilnisky (1980) and Chichilnisky and Heal (1979).

As seen in Theorems 2, 3, and 4, Sections 2 and 3, the graph aggregation rule and the Stolper–Gorman–Samuelson rule, are only well defined under very restrictive conditions on individual preferences. It was shown in Chichilnisky (1980) that in general, no continuous aggregation rule for preferences may exist that is simultaneously anonymous and respects unanimity. In the following we shall prove that the G^* aggregation procedure, which is clearly continuous and anonymous, does not in general respect unanimity: it will be shown to respect unanimity only in a very special case. This therefore agrees with the impossibility result just cited. However, there is a simple transformation of the G^* procedure which does meet all three conditions of continuity, anonymity and respect of unanimity. This aggregation procedure will be shown to be defined only on families of preferences which come within the scope of Theorem 1 of Chichilnisky and Heal (1979), which gives necessary and sufficient conditions on the space of preferences for the existence of a continuous anonymous rule which respects unanimity. In this latter case, the space of preferences is contractible, i.e., it can be continuously deformed into a single preference.

We next turn to a demonstration of the fact that the G^* procedure does not in general respect unanimity. This fact is easily seen. To a point x in X, the G^* procedure assigns a normal identical to the normals of the preferences indexed by $i=1,2,\ldots,M$, at points x_i satisfying

$$\text{(a)} \quad \sum_i x_i = x,$$

and

$$\text{(b)} \quad T_i(x_i) = T_j(x_j) \quad \text{for all } i \text{ and } j.$$

The normal assigned to x is thus not related to the normals of the individual preferences at x, and it follows that these could be identical and yet the outcome at x differs from them. Thus all individual preferences might be identical, yet the social preference would differ from them, so that unanimity is not respected. In fact, one can establish:

Theorem 6. The G^* procedure respects unanimity if and only if preferences are identical and homothetic.

The proof is immediate.

In the case of identical preferences, the space of preferences is a point, clearly contractible, and contractibility of the space of preferences is, according to Theorem 1 of Chichilnisky and Heal (1979), a necessary and sufficient condition for the existence of a continuous anonymous and unanimity-preserving social choice rule when there are finitely many individuals.

We next turn to a modification of the G^* procedure which respects unanimity whenever it is defined, while still being continuous and anonymous. For reasons which will become obvious, we denote this modified procedure by G^*_M, if M is the number of agents. Let $p_1, \ldots, p_M$ be a preference profile within the domain on which G^* is defined, and let p be the outcome assigned to this by G^*, i.e.,

$$G^*(p_1, \ldots, p_M) = p.$$

Let p' be the outcome assigned to the same profile by G^*_M, i.e.,

$$G^*_M(p_1, \ldots, p_M) = p'.$$

Then p' is constructed so that the normal to the indifference surface at any point x in R^n is identical to the normal assigned by p to the point Mx, i.e.,

$$T'(x) = T(Mx).$$

p' is thus constructed by 'pulling back' the preference p toward the origin by a factor $1/M$.

Theorem 7. The G^*_M procedure respects unanimity whenever it yields a preference.

Proof. Consider a profile $p_1, p_2, \ldots, p_M$. Let

$$G^*(p_1, \ldots, p_M) = p'.$$

Now by construction (see the discussion following Lemma 4),

$$T'(x) = T_1(x_1) = T_2(x_2) = \cdots = T_m(x_M) \quad \text{where} \quad \sum_{i=1}^{M} x_i = x.$$

Thus p' assigns to x a normal equal to the common direction given by the M individual preferences to points x_i whose sum is x. So in particular if

$$p_1 = p_2 = p_3 = \cdots = p_M = p,$$

then

$$T'(x)=T\left(\frac{1}{M}x\right). \tag{a}$$

But

$$G^*_M(p_1,\ldots,p_M)=p'',$$

where

$$T''(x)=T'(Mx).$$

Hence,

$$T''\left(\frac{1}{M}x\right)=T'(x),$$

and by (a),

$$T''\left(\frac{1}{M}x\right)=T\left(\frac{1}{M}x\right).$$

Thus if all individual preferences assign the same direction to a point, then G^*_M yields a preference giving that direction. This completes the proof.

The G^*_M procedure is thus a continuous anonymous social choice rule which respects unanimity whenever it is defined. Now Theorem 1 of Chichilnisky and Heal (1979) states that a necessary and sufficient condition for the existence of such a rule, is that the space of preferences is contractible. Intuitively, a contractible space is one which can be continuously deformed through itself into one of its points.

In Theorem 5 above, we showed that, if our attention is restricted to preferences whose tangency paths have a unique point of intersection, then a necessary and sufficient condition for G^* (and so G^*_M) aggregation to yield a preference which is complete, is that it be applied to preferences which are affinely identical and homothetic. But if preferences are affinely identical, it is clear that there is a continuous deformation, indeed a linear translation, which will make them identical. This translation is one which sends every point on a ray, along that ray and into the origin: this, when applied to the translated origins of the preferences, precisely undoes the process by which members of an affinely identical and homothetic family are generated from a single homothetic preference. This translation enables us to deform a space of A.H.I. preferences continuously through itself into one of its points, establishing that it is contractible.

We see from this that, in the case of preferences whose tangency paths have a unique point of intersection, there is indeed a deformation meeting the conditions of Theorem 1 of Chichilnisky and Heal (1979), cited above. The fact that there exists such a deformation that is linear, follows naturally from the fact that the operations involved in defining G^* and G^*_M are entirely linear operations. These are linear social choice rules, and lead to a satisfactory outcome on spaces of preferences with linear tangency paths which are identical up to linear transformations. As in the earlier sections, the case of preferences whose tangency paths do not have unique points of intersection remains to be treated.

Appendix

Theorem 2. If preferences satisfy the conditions of Theorem 1 and are complete, then they are homothetic and identical almost everywhere.

Proof. The sufficient conditions of Theorem 1 are that there exists an open half-space S in R_{2n}, supported by the origin and with RS in its boundary, such that

$$G_i \subset S \quad \text{for all } i.$$

Now, if preferences are complete, then by Lemma 2, for almost every $(x, y) \in R^+_{2n}$, either $(x, y) \in G_i$ or $(y, x) \in G_i$. We shall also show that for almost every $(x, y) \in R^+_{2n}$, either $(x, y) \in S$ or $(y, x) \in S$, where S is the half-space of Theorem 1, represented as

$$S = \{(x, y): p \cdot (x, y) > 0\} \quad \text{for some} \quad p \in R_{2n}.$$

We know that RS is in the boundary of S. Now by definition, $RS = \{(x, y)\colon x = y\}$. Hence, we have

$$p \cdot (x, x) = 0.$$

Therefore, if we partition p into two n-component vectors p_1 and p_2, $p = (p_1, p_2)$,

$$p_1 \cdot x = -p_2 \cdot x \quad \text{for any} \quad x \in R_n.$$

Consider $(x, y) \in S$. It satisfies

$$p \cdot (x, y) > 0, \quad \text{i.e.,} \quad p_1 \cdot x + p_2 \cdot y > 0,$$

by definition of S. Since $p_1 \cdot x = -p_2 \cdot x$ and $p_1 \cdot y = -p_2 \cdot y$, it follows that

$$p \cdot (y, x) = p_1 \cdot y + p_2 \cdot x = -p_1 \cdot x - p_2 \cdot y < 0.$$

So for almost every $(x, y) \in R^+_{2n}$, either $(x, y) \in S$ or $(y, x) \in S$. Since under the conditions postulated, $G_i \subset S, G_i \subset R^+_{2n}$, and for almost every $(x, y) \in R^+_{2n}$, both the following statements are true:

$$(x, y) \in G_i \quad \text{or} \quad (y, x) \in G_i,$$

$$(x, y) \in S \quad \text{or} \quad (y, x) \in S.$$

It follows that G_i and S differ on R^+_{2n} by at most a set of measure zero. For suppose this to be false. Then either there exists an open set U, $U \subset G_i$ and $U \not\subset S$, or there exists an open set U, $U \subset S$ and $U \not\subset G_i$. The first of these is impossible, as $G_i \subset S$. In the latter case, let

$$R(U) = \{(x, y) : (y, x) \in U\}.$$

This is also an open set in R^+_{2n}. It is not contained in S and hence not in G_i. This violates Lemma 2.

We have now established that G_i and S are identical on R^+_{2n} up to a set of measure zero, and hence that all G_i are identical up to a set of measure zero.

Now the intersection of S with R^+_{2n} is a cone with vertex at the origin. Clearly, if the graph G_i of a preference is a cone with vertex at the origin, then the preference is homothetic. For $(x, y) \in G_i$ implies $(\lambda x, \lambda y) \in G_i$ for any positive scalar λ. Hence, x preferred to y implies λx preferred to λy. This, in turn, implies homotheticity. To see this, consider a sequence $x_n \in R_n$, satisfying

$$\lim_{n \to \infty} x_n = x, \quad x_n p_i x, \quad x i_i y.$$

Then,

$$\lim_{n \to \infty} \lambda x_n = \lambda x, \quad \lambda x_n p_i \lambda x, \quad \lambda > 0.$$

We have to show that $\lambda x i_i \lambda y$. Suppose not. Then either $\lambda x p_i \lambda y$ or $\lambda y p_i \lambda x$. In the first case,

$$(\lambda x, \lambda y) \in G_i, \quad \text{but} \quad (x, y) \notin G_i.$$

In the second case,

$$(\lambda y, \lambda x) \in G_i, \quad \text{but} \quad (y, x) \notin G_i.$$

In both cases, we have a contradiction.

We have now shown that the graphs of the preferences are identical up to sets of measure zero, and equal, up to a set of measure zero, to the graph of a homothetic preference. This completes the proof.

Lemma 5. For each $i=1,\dots,M$, *let* $y_i=f_i(t_i)$, $y_i\in R_{n-1}$, $t_i\in R_1$, *be the equation of a smooth curve, and its graph be* $C_i=\{(t,y): y_i=f_i(t_i)\}$ *in* R_n. *Consider* $\sum_{i=1}^M C_i$. *A necessary and sufficient condition for there to exist a curve* $f: R_1\to R_{n-1}$ *with* $y=f(t)$ *such that*

$$\sum_{i=1}^{M} C_i=\{(t,y): y=f(t)\},$$

is that the f_i *are of the form* $y_i=\alpha_i+\beta t_i$ *with* $\alpha_i,\beta\in R_{n-1}$.

Proof. Sufficiency follows immediately from the fact that

$$\sum_{i=1}^{M} C_i=\left\{(t,y): y=\sum_i \alpha_i+\beta t\right\}.$$

To establish necessity, note that

$$\sum_i C_i=\left\{(t,y):(t,y)=\sum_i (t_i, f_i(t_i))\right\}.$$

In general, this can be interpreted as the graph of a function $H: R_m\to R_{n-1}$, because $f(t_i)$ depends on t_i (and not on $\sum_i t_i$), and assumes a different value for each M-tuple $(t_1,\dots,t_M)$. However, what is required is the existence of another function, $f: R_1\to R_{n-1}$ such that

$$\sum_i C_i=\{(t,y): y=f(t)\},$$

namely, that the function whose values are $\sum_i (f_i(t_i))$ actually depends on $\sum_i t_i$ only, i.e.,

$$\sum_i f_i(t_i)=f\left(\sum_i t_i\right)=\sum_i f_i(t_i') \quad \text{whenever} \quad \sum_i t_i=\sum_i t_i'.$$

If this were the case, then for any increment $\varepsilon>0$,

$$f_1(t_1+\varepsilon)+f_2(t_2-\varepsilon)+\sum_{j>2} f_j(t_j)=\sum_{i\geq 1} f_i(t_i),$$

since

$$(t_1+\varepsilon)+(t_2-\varepsilon)=t_1+t_2.$$

This implies, in particular, that

$$Df_1(t_1)=Df_2(t_2) \quad \text{for all } t_1 \text{ and } t_2, \tag{*}$$

satisfying $0 \leqq t_1+t_2 \leqq \sum_i t_i$. In particular, when $t_1=t_2$,

$$Df_1(t_1)=Df_2(t_1), \tag{**}$$

so that from (*) and (**),

$$Df_1(t_1)=Df_2(t_2)=Df_1(t_2).$$

Therefore, Df_1 is a constant over the relevant range. Since the same argument is valid for all the f_i, they are linear with identical gradients. This completes the proof.

Lemma 5 gives us necessary and sufficient conditions for the sum of curves to be a curve. Therefore, it enables us to state immediately a condition which is necessary and sufficient for the closure of the graph of the aggregate relation, restricted to a given tangent, to have an intersection with the reflexive space RS which defines a curve. This is, in turn, a necessary condition for $\sum(\overline{G_i/T}) \cap RS$ to define a non-intersecting family of curves. The condition is that for each i, $(\overline{G_i/T}) \cap RS$ is of the form

$$y_i=\alpha_i(T)+\beta(T)t_i, \qquad y_i \in R_{n-1}, \quad t_i \in R_1.$$

The following lemma gives conditions sufficient to ensure that the aggregate family of curves is non-intersecting, so that the tangency paths of the aggregate relation are non-intersecting in the choice space.

Lemma 6. Consider M families of curves in R_n defined by

$$y_i=\alpha_i(T)+\beta(T)t_i, \qquad y_i \in R_{n-1}, \quad t_i \in R_1,$$

where i is the index of the family and T is a parameter in R_n. Each family $i=1,2,\dots,M$ satisfies:

(a) *there is one member of the family through every point of R_n^+,*
(b) *any point of intersection of two members of a family corresponding to different T values, will have all of its components non-positive.*

Then no two members of the aggregate family

$$y=\sum_i \alpha_i(T)+\beta(T)t$$

will intersect in R_n^+.

Proof. Consider

$$y_i=\alpha_i(T)+\beta(T)t_i, \qquad y_i=\alpha_i(T')+\beta(T')t_i,$$

for $T\neq T'$. At an intersection, the RHS are equal, so t_i satisfies

$$(\beta(T')-\beta(T))t_i=\alpha_i(T)-\alpha_i(T'). \tag{A}$$

Consider a t_i satisfying (A). Recall that $t_i\in R_1$. It must be non-positive by condition (b) of the lemma. Now consider the aggregate family

$$y=\sum_i \alpha_i(T)+\beta(T)t, \qquad y=\sum_i \alpha_i(T')+\beta(T')t,$$

and consider the set of t for which the RHS are equal,

$$(\beta(T')-\beta(T))t=\sum_i(\alpha_i(T)-\alpha_i(T')).$$

As each element of the sum on the right is a non-positive scalar multiple ‹ $(\beta(T')-\beta(T))$, the result follows.

The lemma shows that, given complete preferences with $\sum_i (\overline{G_i/T})\cap RS$ ‹ the appropriate forms, condition (b) is sufficient for non-intersection of th aggregate family. It is also necessary in the sense that if this condition is n imposed, it is always possible to find a pair of complete families with ‹ intersection in R_n^+ whose sum nevertheless is not free of intersections. Fig illustrates this. The aggregate of this family therefore cannot be a transiti relation.

It should be noted that condition (b) of Lemma 6 imposes a monotonici property on preferences. Condition (b) requires that the intersection of a two tangency paths lies in the nonpositive orthant. In this case, any tangen path in the positive orthant must be non-decreasing in every dimension. that preferences are monotone in the following sense:

$$a\geq b\rightarrow a p_i b \quad \text{for any } i.$$

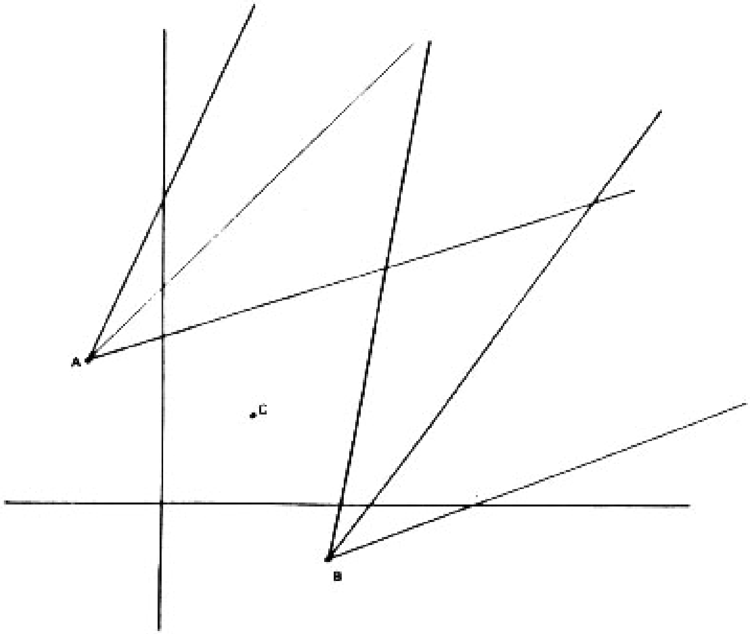

Fig. 4. Two families with intersections at A and B, their sum has intersections at $C = A + B$.

We now have the necessary and sufficient conditions for

$$\sum_{i=1}^{M} \overline{(G_i/T)} \cap RS$$

to be a non-intersecting family of curves in RS. This non-intersecting property implies that at any point there is a unique most-preferred direction, and hence, given smoothness, a unique indifference curve. Together with irreflexivity of the strict order, this implies local integrability and local transitivity. However, as noted above, condition (b) of Lemma 6 implies that preferences are monotone, which implies that local integrability is equivalent to global integrability [see Debreu (1972)]. The conditions of Lemma 6 are therefore necessary and sufficient for transitivity of the aggregate relation.

The conditions we have discussed so far can readily be transformed into more familiar conditions on preferences. We have investigated conditions on paths in RS which link points with a common tangent. RS is a space of dimension n and can be projected by taking its first n components onto the choice space R_n: the projections of these paths then become tangency paths in R_n, i.e., paths linking choices at given prices but different income levels.

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