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THE TOPOLOGICAL EQUIVALENCE OF THE PARETO CONDITION AND THE EXISTENCE OF A DICTATOR

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The paper studies two standard properties of rules for aggregating individual into social preferences: non-dictatorship and the Pareto condition. Together with the condition of independence of irrelevant alternatives, these are the three basic axioms of Arrow's social choice paradox.

We prove the topological equivalence between the Pareto condition and the existence of a dictator for continuous rules. The axiom of independence of irrelevant alternatives is not required.

The results use a topological framework for aggregation introduced in Chichilnisky (1980), but under different conditions. In Chichilnisky (1980) rules are anonymous and respect unanimity. Since anonymity is strictly stronger than the condition of non-dictatorship, while respect of unanimity is strictly weaker than the Pareto condition, the two sets of conditions are not comparable.

1. Introduction

Arrow's social choice paradox exhibits a contradiction between three conditions on the aggregation of individual preferences: non-dictatorship, the Pareto condition, and that of independence of irrelevant alternatives.

Recent work has proven the existence of social choice paradoxes for continuous aggregation rules without requiring independence of irrelevant alternatives. In Chichilnisky (1980) the rules are required to be anonymous and respect unanimity, whereas in Chichilnisky (1981) decisive majority conditions are required. While these results exhibit the topological nature of social choice paradoxes, they are proven under rather different assumptions from Arrow's. For instance, the condition of respect of unanimity is strictly weaker than the Pareto condition, while that of anonymity is stronger than non-dictatorship, so that the results obtained under one set of conditions are not comparable to those obtained under the other. In addition, although the

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independence axiom is not assumed in this recent work, the aggregation rules are required to be continuous.

The purpose of this paper is to show that two of Arrow's conditions, Pareto and non-dictatorship, are topologically inconsistent with each other, in the sense that any continuous social choice rule that satisfies the Pareto condition can be continuously deformed into a dictatorial rule. The independence axiom is not needed to prove this result. With more than two voters a weak form of positive association is required on the rule.

The paper is organised as follows: Section 2 contains notations and definitions. Section 3 proves the result for a special case which admits a simple geometrical proof: when there are two voters and their preferences are linear. Section 4 proves the topological equivalence between the Pareto condition and the existence of a dictator, with any number of voters and unrestricted domains of preferences, when the rule satisfies a weak form of positive association.

2. Notation and definitions

Let X denote the *choice space*. X will be assumed to be C^2 diffeomorphic[1] to the closed unit ball B^n in R^n.

A *preference* p is a C^1 vector field[2] on X, $p:X \to R^n$, which is locally the gradient field of a utility function u on X. The vector $p(x)$ at the choice x in X is therefore the normal to the tangent space of the indifference surface of u at x; see fig. 1. Following a standard convention in social choice theory, preferences are *ordinal*, so that the vector field $p(x)$ is normalized[3] to be of unit length, i.e., $\|p(x)\| = 1$ for all x in X.

The *space of preferences* on X denoted P is a subset of the Banach space of all C^1 vector fields on X, $V(X)$, and is characterized within $V(X)$ by the normalization and the Frobenius integrability conditions; see e.g. Debreu (1972). $V(X)$ is a Banach space when endowed with the C^1 sup norm.

Since both Frobenius conditions and the normalisation are closed conditions in the C^1 sup norm, the space P endowed with the topology inherited from $V(X)$ is a complete space. The space $V(X)$ is infinite dimensional, and P contains infinite dimensional manifolds; see Chichilnisky (1976).

We assume there are k agents ($k \geqq 2$). A *profile* $p_1, \ldots, p_k$ is an ordered k-tuple of preferences of the voters, so that the *space of profiles* is P^k.

[1] Let $X \subset R^n$, $Y \subset R^m$ be smooth manifolds with boundary. $f:X \to Y$ is C^2 if it admits an extension to a twice continuously differentiable map defined on a neighbourhood of X. X is C^2 diffeomorphic to Y when there exists a C^2 one-to-one and onto map $f:X \to Y$ whose inverse is also C^2.

[2] A vector field $v(x)$ on X is C^1 if it admits an extension to a C^1 vector field defined on a neighbourhood of X.

[3] This implies that v is not satiated in the interior of X, X^0. Related results can be obtained when preferences admit satiation in X^0; see Chichilnisky (1979).

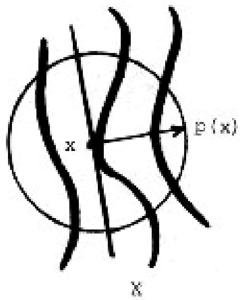

Fig. 1. $p(x)$ is the normalized gradient of the preference p at the choice x.

A *social aggregation rule* is a map that assigns to each profile in P^k a social preference in P, $\phi: P^k \to P$.

A rule ϕ is said to satisfy a *Pareto condition*, or to be Pareto, when the following is always true: if for all voters $1, \ldots, k$, a choice x is preferred to another y (for instance if the utility functions that represent $p_1, \ldots, p_k$ give a higher value to x than to y) then the social preference $\phi(p_1, \ldots, p_k)$ prefers x to y.

A rule ϕ satisfies the *weak positive association* condition if $\phi(p_1, \ldots, p_i, \ldots, p_k) = -p_i$ for some $i = 1, \ldots, k$ and some $(p_1, \ldots, p_k) \in P^k$, implies

$$\phi(-p_i, \ldots, -p_i, p_i, -p_i, \ldots, p_i) \neq p_i.$$

A Pareto rule satisfying the weak positive association condition is denoted a *W-Pareto rule*.

A rule ϕ *respects unanimity* when the restriction of the map ϕ on the set $D = \{(p_1, \ldots, p_k): p_i = p_j, \forall i, j = 1, \ldots, k\}$, is the identity map on D, i.e., $\phi/D = id_D$. Note that the Pareto condition implies respect of unanimity. A preference is called *linear* when it is induced by a linear utility function on X.

An aggregation rule ϕ_d is *dictatorial* (with dictator d) if ϕ_d is the projection onto the dth coordinate, $\phi_d(p_1, \ldots, p_k) = p_d$, $\forall (p_1, \ldots, p_k) \in P^k$.

Let f and g be two continuous functions between two topological spaces Y and Z. f is said to be *homotopic* to g, or a *continuous deformation* of g, when there exists a continuous map

$$\Pi: Y \times [0, 1] \to Z,$$

satisfying $\Pi(y, 0) = f(y)$ and $\Pi(y, 1) = g(y)$, for all y in Y.

3. An example with two voters and linear preferences

We now study a particular case, where there are two voters and their preferences are linear. This case admits a simpler geometrical proof that the

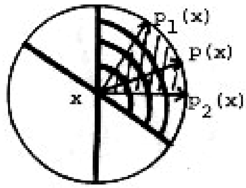

Fig. 2. The Pareto property for two voters and a two-dimensional choice space. The area shaded with circular lines is the dual cone of p_1 and p_2, the area doubly shaded is the cone determined by p_1 and p_2.

Pareto condition is topologically equivalent to the existence of a dictator.

Let $\bar{P}$ denote the space of linear preferences on the choice space $X \subset R^{n-1}$, $n \geqq 1$. Each $p \in \bar{P}$ can be identified by *one* vector, the gradient of the linear utility function corresponding to p. Since gradients are normalised to be of unit length, the space of all linear preferences on X can therefore be identified with the space of their gradients, so that $\bar{P}$ is the nth sphere in R^{n+1}, i.e.,

$$\bar{P} = S^n = \{v \in R^{n+1} : \|v\| = 1\}.$$

Therefore with 2 voters the space of profiles $\bar{P}^2$ is here $(S^n)^2$. When $n = 1$, $\bar{P}^2 = S^1 \times S^1 = T^2$, the two-dimensional torus. For any $n \geqq 1$, a social choice rule is here a map $\phi : (S^n)^2 \to S^n$.

Given two preferences p_1, p_2 in S^n, the Pareto property implies that the function ϕ maps the profile (p_1, p_2) into a preference $p = \phi(p_1, p_2)$ which is contained in the cone determined by p_1 and p_2 in S^n, as illustrated in fig. 2. This is because the Pareto condition implies that p must have a positive inner product with all vectors of the 'dual cone' of (p_1, p_2), i.e., with all vectors in the set

$$\{g \in S^n : g \cdot p_1 \geqq 0 \quad \text{and} \quad g \cdot p_2 \geqq 0\}.$$

Thus, if ϕ is Pareto, $\phi(p_1, p_2)$ must belong to the dual of this dual cone, i.e., p must belong to the cone determined by p_1 and p_2. This property is of course trivially satisfied by any $q \in S^n$ when p_1 and p_2 are exactly opposed to each other, i.e., when $p_1 = -p_2$ in S^n.

Fig. 3 shows an example of a Pareto rule for two voters, and a two-dimensional choice space. The Pareto property implies respect of unanimity, i.e., $\phi(p, p) = p$ for all $p \in S^1$. In particular, if a rule ϕ is Pareto, for each $p \in S^1$, the inverse image of p under ϕ, $\phi^{-1}(p) \subset S^1 \times S^1$ intersects the set

$$D = \{(p_1, p_2) \in (S^1)^2 : p_1 = p_2\},$$

exactly at one point, $\phi^{-1}(p) \cap D = \{p, p\}$, $\forall p \in S^1$.

Fig. 4 represents a dictatorial rule: clearly, for each $p \in S^1$, $\phi^{-1}(p) \cap D = \{p, p\}$ in this case as well. Note that the surfaces $\phi^{-1}(p) (\forall p \in S^1)$ of the

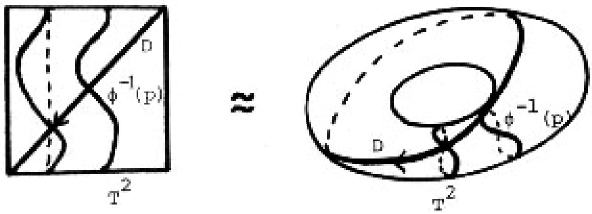

Fig. 3. The space of profiles of two voters' preferences for two-dimensional choices and linear preferences is the torus $T^2=(S^1)^2$. The curved lines indicate the surfaces $\phi^{-1}(p)$ corresponding to a fixed value $p\in\bar{P}$ of the map $\phi:\bar{P}^2\to\bar{P}$. The map ϕ is Pareto; therefore its indifference surfaces intersect D exactly once.

Pareto rule in fig. 3, can be continuously deformed into those of the dictatorial rule in fig. 4. This is an illustration of the following result.

Fig. 5 shows an example of a rule that is clearly *not* Pareto, since each $\phi^{-1}(p)$ intersects D twice: the following theorem will prove that this rule is not topologically equivalent to a dictatorial rule either.

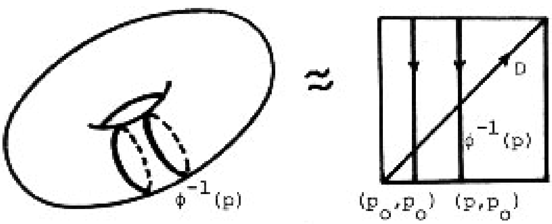

Fig. 4. A dictatorial rule with two agents and two commodities.

Theorem 1. Let $\phi:\bar{P}^2\to\bar{P}$ be a continuous Pareto rule. Then ϕ is homotopic to a dictatorial rule.

Proof. Let p_0 be an arbitrary preference in $\bar{P}$, and let $p_1=p_0$, $p_2=-p_0$. Continuity and the Pareto property imply that either

$$\phi(p_1,p_2)=p_0,\quad \text{or else}\quad \phi(p_1,p_2)=-p_0.$$

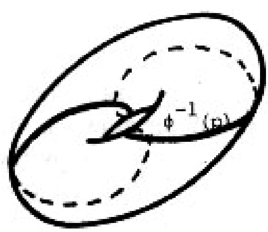

Fig. 5. Each surface $\phi^{-1}(p)$ intersects the diagonal D exactly two times: the map ϕ is *not* Pareto. Therefore it is not topologically equivalent to a dictatorial rule either.

Assume, without loss of generality, that

$$\phi(p_1, p_2) = p_0.$$

Continuity then implies that for all other p in S^n, $\phi(p, -p) = p$. i.e., whenever agent one has a preference opposed to that of agent two, then agent one is a dictator. Therefore for any *other* pair of preferences q_1 and q_2, $\phi(q_1, q_2) \neq -q_1$, because by the Pareto condition, either $q_1 \neq -q_2$ and $\phi(q_1, q_2)$ is in the cone generated by q_1 and q_2 (and $-q_1$ does not satisfy this), or else $q_1 = -q_2$ in which case $\phi(q_1, q_2) = q_1$ as shown above.

Define now the map

$$H: (S^n)^2 \times [0, 1] \to S^n,$$

by

$$H(p_1, p_2, t) = \frac{tp_1 + (1-t)\phi(p_1, p_2)}{\|tp_1 + (1-t)\phi(p_1, p_2)\|}.$$

This map is always well defined because $\phi(p_1, p_2) \neq -p_1$ so that the denominator is never zero. Since for all $(p_1, p_2) \in (S^n)^2$,

$$H(p_1, p_2, 0) = \phi(p_1, p_2) \quad \text{and} \quad H(p_1, p_2, 1) = p_1,$$

H is the desired homotopy between the rule ϕ and a dictatorial rule on $(S^n)^2$.

4. The topological equivalence of the Pareto condition and the existence of a dictator

We now study the case of an unrestricted domain of smooth preferences P as defined on section 2, defined on an $(n+1)$-dimensional choice space, $n \geqq 2$, and with k voters, $k \geqq 2$. For further references on algebraic topology, see e.g. Spanier (1963).

Theorem 2. Let $\phi: P^k \to P$ be a continuous W-Pareto aggregation rule. Then ϕ is homotopic to a dictatorial rule.

Proof. Let x be a choice in X, and let S^n be the unit sphere in R^{n+1}. Given a chart for X at x, each preference p in P determines uniquely a point z in S^n, the unit vector normal to the indifference surface of p at x in the orienting direction $p(x)$. This determines a map Γ from the space of preferences P into S^n, which is continuous by the choice of topology in P. The map Γ can be chosen to be continuous and *onto* S^n, i.e., for all z in S^n there exists some preference p on X that has z as a normal vector at x, i.e. $\Gamma(p) = z$. We can also define a continuous map $\lambda: S^n \to P$ assigning for each z in S^n a preference on X with preferred direction equal to z at the choice

x. $\lambda(z)$ can be chosen as a linear preference with indifference surfaces orthogonal to the vector z.

$$\begin{array}{ccc} & P & \\ {\scriptstyle\lambda}\nearrow & & \searrow{\scriptstyle\Gamma} \\ S^n & \xrightarrow{I} & S^n \end{array} \tag{1}$$

The map I defined by making the above diagram commutative, i.e., $I = \Gamma \circ \lambda$, is the identity map on S^n.

Let $\phi : P^k \to P$ be a continuous rule satisfying the Pareto condition. Then, for each x in X we can define the map ψ by the diagram

$$\begin{array}{ccc} P^k & \xrightarrow{\phi} & P \\ {\scriptstyle\lambda^k}\downarrow & & \downarrow{\scriptstyle\Gamma} \\ (S^n)^k & \xrightarrow{\psi} & S^n \end{array}$$

i.e., $\psi(z_1, \ldots, z_k) = \Gamma(\phi(\lambda(z_1), \ldots, \lambda(z_k)))$ for all $(z_1, \ldots, z_k) \in (S^n)^k$. ψ is continuous, being the composition of continuous maps.

For a given $z_0 \in S^n$, define

$$G_i = \{(z_1, \ldots, z_k) \in (S^n)^k \text{ such that } z_j = z_0, \forall j \neq i\}.$$

Let $C(z_1, z_2)$ denote a shortest circular segment of a great circle on S^n containing both z_1 and z_2, which is uniquely defined when $z_1 \neq z_2$. Since the rule ϕ satisfies the Pareto condition, for any $i = 1, \ldots, k$, the restriction map ψ/G_i must satisfy

$$\psi/G_i(z_1, \ldots, z_k) \in C(z_i, z_0) \quad \text{when} \quad z_i \neq z_0,$$

because $\psi/G_i(z_1, \ldots, z_k)$ must have positive inner product with all vectors in S^n which have positive inner product with both z_0 and z_i. Since each G_i is homeomorphic to a sphere S^n, we can define the degree of the map

$$\psi/G_i : G_i \to S_n,$$

and by the Pareto condition,

$$\deg(\psi/G_i) \text{ is either } 0 \text{ or } 1 \text{ for all } i = 1, \ldots, k. \tag{1}$$

Now, since ϕ is Pareto, ψ satisfies the respect of unanimity condition, i.e.,

$$\psi/D = id_D. \tag{2}$$

Since D is homeomorphic to S^n, we can define the degree of the restriction map $\psi/D : D \to S^n$, and (2) implies

$$\deg(\psi/D) = 1. \tag{3}$$

Now, let $\Pi_n((S^n)^k)$ be the nth homotopy group of $(S^n)^k$. Then

$$\Pi_n((S^n)^k) \approx \bigoplus_{i=1}^{k} \Pi_n(S^n); \tag{4}$$

see e.g. Spanier (1963, p. 419, B.5).[4]

Consider now the inclusion map

$$in_D : S^n \to ((S^n)^k),$$

defined by

$$in_D(z) = (z, \ldots, z) \in D \subset (S^n)^k \quad \text{for any} \quad z \in S^n.$$

Similarly for all $i = 1, \ldots, k$ define

$$in_{G_i} : S^n \to (S^n)^k$$

by

$$in_{G_i}(z) = (z_0, \ldots, \overset{i\text{th place}}{z}, \ldots, z_0) \in G_i \subset (S^n)^k \quad \text{for any} \quad z \in S^n.$$

Let $P_i : (S^n)^k \to S^n$ be the projection map onto the ith coordinate. Then for each i, the two composition maps:

$$P_i \circ in_D : S^n \to S^n \quad \text{and} \quad P_i \circ in_{G_i} : S^n \to S^n,$$

are both the identity map on S^n. In particular the homotopy classes of both these maps are that of $id : S^n \to S^n$, i.e.,

$$[P_i \circ in_D] = [P_i \circ in_{G_i}] = [id]. \tag{5}$$

Let $[in_{G_i}]$ denote the homotopy class of $in_{G_i} : S^n \to (S^n)^k$ in $\Pi_n((S^n)^k)$, and let the map $T: S^n \to (S^n)^k$ be such that

$$[T] = \sum_{i=1}^{k} [in_{G_i}] \in \Pi_n((S^n)^k). \tag{6}$$

We can take T to satisfy

$$T(S^n) \subset \bigcup_{i=1}^{k} G_i \subset (S^n)^k,$$

and

$$[P_i \circ T] = [P_i \circ in_{G_i}] \in \Pi_n(S^n), \tag{7}$$

for each $i = 1, \ldots, k$.

Therefore by (5) and (7) for each $i = 1, \ldots, k$,

$$[P_i \circ T] = [P_i \circ in_D] = [id], \tag{8}$$

[4] $\Pi_n((S^n)^k)$ is written as $\oplus_{i=1}^k \Pi_n(S^n)$ instead of as $\times_{i=1}^k \Pi_n S^n)$ to indicate that it is an abelian group, since $n \geq 2$.

i.e., $P_i^*[T]=P_i^*[in_D]$, where P_i^* is the map induced by P_i at the homotopy level, $P_i^*:\Pi_n((S^n)^k)\to\Pi_n(S^n)$. Since (8) is true for all $i=1,\dots,k$, (4) implies

$$[T]=[in_D]. \tag{9}$$

[see also Chichilnisky (1981, lemma 1)]. Therefore

$$[\psi\circ T]=[\psi\circ in_D], \tag{10}$$

and by (6) this implies

$$[\psi\circ T]=\left[\sum_{i=1}^{k}\psi\circ in_{G_i}\right]=[\psi\circ in_D], \tag{11}$$

so that

$$\deg(\psi\circ T)=\sum_{i=1}^{k}\deg(\psi\circ in_{G_i})=\deg(\psi\circ in_D).$$

Since $\deg(\psi\circ in_D)=\deg(\psi/D)$, and by (3) $\deg(\psi/D)=1$, it follows that

$$\sum_{i=1}^{k}\deg(\psi\circ in_{G_i})=1. \tag{12}$$

Since $\deg(\psi\circ in_{G_i})=\deg(\psi/G_i)$, from (1) and (12) we obtain

$$\deg(\psi/G_d)=1 \quad \text{for some} \quad d\in\{1,\dots,k\}, \tag{13}$$

and

$$\deg(\psi/G_i)=0 \quad \text{for all} \quad i\neq d. \tag{14}$$

Now, the Pareto property and the continuity of ψ imply that for all $i=1,\dots,k$,

$$\psi\circ in_{G_i}(-z_0)=\psi(z_0,\dots,\overbrace{-z_0,}^{\text{ith place}}\dots,z_0)\in\{z_0,-z_0\}. \tag{15}$$

This is because the Pareto condition implies

$$\psi\circ in_{G_i}(z)\in C(z_0,z) \quad \text{for } any \quad z\in S^n. \tag{16}$$

It follows also from (16) that if $\psi\circ in_{G_i}(-z_0)=z_0$, then the value $-z_0$ is never assumed by the map ψ/G_i. Therefore for any $i=1,\dots,k$,

$$\psi\circ in_{G_i}(-z_0)=z_0 \quad \text{implies} \quad \deg(\psi/G_i)=0. \tag{17}$$

By (12), therefore, $\psi\circ in_{G_j}(-z_0)=-z_0$ for some $j=1,\dots,k$. Now by (16) if

$$\psi\circ in_{G_j}(-z_0)=-z_0,$$

then

$$\deg(\psi \circ in_{G_i}) > 0, \tag{18}$$

so that $\deg(\psi \circ in_{G_i}) = 1$ by (1).

It follows from (13) and (14) that for some $d = 1, \ldots, k$,

$$\psi \circ in_{G_d}(-z_0) = -z_0, \tag{19}$$

and

$$\psi \circ in_{G_i}(-z_0) = z_0 \quad \text{for all} \quad i \neq d. \tag{20}$$

In particular, $\psi(z_1, \ldots, z_k) \neq -z_d$ for all profile $(z_1, \ldots, z_k)$ in G_i, all $i = 1, \ldots, k$.

Note that in the argument given above the vector z_0 is arbitrary chosen. Therefore, by continuity of ψ, for any $z_0^1 \in S^n$,

$$(z_1, \ldots, z_k) \neq -z_d, \tag{21}$$

whenever $(z_1, \ldots, z_k)$ is in $\bigcup_i G_i(z_0^1)$, where

$$G_i(z_0^1) = \{(z_1, \ldots, z_k) \in (S^n)^k : z_i = z_0^1, \forall i \neq d\}.$$

Without loss of generality, assume $d = 1$. Then (21) implies, in particular, that for any $z_1 \in S^n$,

$$\psi(z_1, -z_1, \ldots, -z_1) = z_1. \tag{22}$$

Consider now any profile $(z_1, \ldots, z_k) \in (S^n)^k$. Since ϕ satisfies the weak positive association condition, if

$$\psi(z_1, \ldots, z_k) = -z_1,$$

it would follow that

$$\psi(z_1, -z_1, \ldots, -z_1) \neq z_1,$$

contradicting (22). Therefore, we have proven in particular that

$$\psi(z_1, \ldots, z_k) \neq -z_1,$$

for all $(z_1, \ldots, z_k) \in (S^n)^k$.

We now return to the map $\phi : P^k \to P$. Since ϕ satisfies both the Pareto and weak positive association conditions, then for each choice $x \in X$, there exists one $d = 1, \ldots, k$ with $\phi(p_1, \ldots, p_k)(x) \neq -p_d(x)$. This is because both the Pareto and the weak positive association condition hold locally, for any x in X; i.e., if y is preferred to x by $p_1, \ldots, p_k$, and y is sufficiently close to x, then it follows that

$$p_1(x)(y-x) \geqq 0 \cdots p_k(x)(y-x) \geqq 0 \quad \text{implies} \quad p(x)(y-x) \geqq 0,$$

which is the Pareto condition for vectors at the sphere S^n centred at x.

The first part of the proof implies therefore that for any x in X there exists some $d = d(x)$ such that

$$\phi(-p_d, \ldots, \overset{d\text{th place}}{\widehat{-p_d}}, p_d, -p_d, \ldots, -p_d)(x) = p_d(x). \tag{23}$$

By continuity of ϕ, $d(x)$ must be the same for all x in X.

In view of the Pareto and the weak positive association conditions, it follows as proven above that for any $p_1, \ldots, p_k \in P^k$,

$$(p_1, \ldots, p_k)(x) \neq -p_d(x). \tag{24}$$

Therefore, for any $x \in X$, $t \in [0, 1]$ and $(p_1, \ldots, p_k) \in P^k$,

$$tp_d(x) + (1-t)\phi(p_1, \ldots, p_k)(x) \neq 0.$$

We can then define the map H on $P^k \times [0, 1]$ by

$$H(p_1, \ldots, p_k, t)(x) = \frac{tp_d(x) + (1-t)\phi(p_1, \ldots, p_k)(x)}{\|tp_d(x) + (1-t)\phi(p_1, \ldots, p_k)(x)\|}.$$

Since $p_d(x)$ is the gradient of the map $p_d(x) \cdot x$, it follows that for each $(p_1, \ldots, p_k, t) \in P^k \times [0, 1]$, $H(p_1, \ldots, p_k, t)$ is in P, being at each $x \in X$ a scalar multiple of a C^1 gradient field on X.

Since for all $x \in X$,

$$H(p_1, \ldots, p_k, 0)(x) = \phi(p_1, \ldots, p_k)(x),$$

and

$$H(p_1, \ldots, p_k, 1)(x) = p_d(x),$$

H is the desired homotopy between ϕ and a dictatorial rule. This completes the proof.

References

Arrow, K., 1963, Social choice and individual values, Cowles Foundation for research in economics monograph 12 (Yale University, New Haven, CT).

Chichilnisky, G., 1976, Manifolds of preferences and equilibria, Report no. 27, Project on efficiency of decision making in economic systems (Harvard University, Cambridge, MA).

Chichilnisky, G., 1980, Social choice and the topology of spaces of preferences, Advances in Mathematics.

Chichilnisky, G., 1981, Structural instability of decisive majority rules, Journal of Mathematics Economics 9, 207–221.

Debreu, G., 1972, Smooth preferences, Econometrica.

Spanier, E.H., 1963, Algebraic topology, McGraw-Hill series in higher mathematics (McGraw-Hill, New York).