SOCIAL AGGREGATION RULES AND CONTINUITY*

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It is shown that any continuous social aggregation rule for smooth preferences cannot simultaneously satisfy the properties of anonymity and respect of unanimity. This is true even when all individual preferences are linear. The relationship between the conditions on the social rule studied here and those of Arrow's paradox is discussed. The first result requires that the normalized gradient of the social choice rule be definable in the interior of the choice space, thus indicating a direction of increase of social preference. A second impossibility result extends the first to cases where the gradient of the social preference may vanish in the interior of the choice space.

I. INTRODUCTION

The vast literature on Social Choice makes little use of the continuity properties of the social aggregation rule.[1] This may be due to the fact that following Arrow's [1963] pioneering work, most existing work has focused on problems where the individuals face a finite number of choices, and is therefore formulated in a combinatorial fashion. However, while problems with finite choices seem in principle simpler to analyze, in effect they may require the use of techniques that limit our intuitive geometric understanding.

In this paper we study instead problems of aggregation with preferences defined on Euclidean choice spaces; preferences are ordinal (i.e., no intensities are registered) and are here assumed to be continuous and, furthermore, smooth. In this context the continuity of the aggregation rule that assigns a social preference to the individual preferences is definable, and can be argued to be a natural property. One reason for requiring continuity is that it is desirable for the social rule to be relatively insensitive to small changes in individual preferences. This makes mistakes in identifying preferences less crucial. It also permits one to approximate social preferences on the basis of a sample of individual preferences.

The main result in Section II has been proved in Chichilnisky [1980] using algebraic topology, and one purpose of this paper is expository: to present and discuss this result in relation to existing ones

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1. Other works where continuity properties are mentioned in a different context are the following: Kelly [1971], Saposnik [1975], and McManus [1977].

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in the literature. It is shown there that two properties of continuous aggregation rules, anonymity (symmetry with respect to voters) and respect of unanimity (unanimity of individual preferences over all choices is mirrored in social preferences) are mutually inconsistent. Particular examples are discussed.

Section III discusses the relationship of our results and conditions with others in the literature—in particular with Arrow's. Section IV proves a new impossibility result, for cases where social preferences may have an undefined gradient. This latter result therefore extends the previous one, which is valid only for rules with nonvanishing gradients in the interior of the choice space, but requires a stronger condition than respect of unanimity, i.e., the rule must satisfy a Pareto condition.

The results indicate that certain basic problems of aggregation derive from the underlying global topological structure of the spaces of preferences.

II. An Impossibility Theorem for Continuous Aggregation Rules That Respect Anonymity and Unanimity

The social choice problems considered here are akin to those of selection of a vector of public goods, and therefore, lend themselves naturally to representation in Euclidean space. We consider here a *choice space* X, which is contained in the positive orthant of n-Euclidean space, i.e., in the set of positive n-dimensional vectors denoted by R^{n+}. X is here a cube in R^{n+}, but more general choice spaces in Euclidean spaces could also be chosen without affecting the results. A *preference* p is identified by assigning to each choice x in X, a vector $p(x)$, which is the gradient of a (differentiable) utility function defined on the choice space. This direction is that of the normal to the tangent plane of the indifference surface through the choice x; it can be thought of as a "most desirable" direction, since it is the direction of the largest increase in utility. The preferences considered here are integrable and smooth, i.e., they are representable by smooth utility functions on the choice space (see e.g., Chichilnisky [1976]).

A preference p is therefore represented by a *vector field* on X, which assigns to each choice x in X a gradient vector $p(x)$. Since no intensities are considered, all vectors are of the same length, say of unit length. The *space of preferences* P is then defined as the set of all C^1 (continuously differentiable) integrable unit vector fields defined on the choice space X. Figure I represents such a preference in

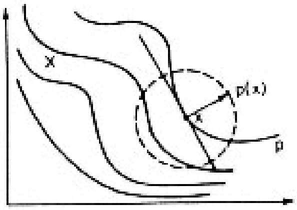

FIGURE I

a two-dimensional choice space. We assume there are k individuals. A *social aggregation rule* for individual preferences is a function that assigns to each k-tuple of individual preferences called a *profile*, another preference in P, called a *social preference*. The condition that the social preferences be smooth integrable vector fields is not essential to the results given below, but it simplifies the exposition. What is required for the impossibility results given here is that the social preference defines at each choice a "most desirable" direction in a continuous manner. In particular, since each vector is normalized so as to be of unit length, this implies that the gradient vector field before normalization does not vanish in the interior of the choice space, since otherwise the normalized gradient is left undefined. The role of nonvanishing condition on vector fields is discussed in the examples give below. This condition is relaxed in Section IV, where preferences with vanishing gradients are studied.

In symbols, a social choice rule is a function from k-tuples of individual preferences into social preferences,

$$\phi : P^k \rightarrow P.$$

In order to define continuity of ϕ, we define convergence in the space of preferences. A sequence $\{p^j\}$ converges to a preference p when the preferred directions of the p^j's converge uniformly to those of the preference p, i.e.,

$$(1^*) \qquad \sup_{x \in X} \| p(x) - p^j(x) \| \rightarrow 0.$$

This induces a corresponding product topology[2] on the space of profiles P^k.

2. With this topology, P will not be a complete topological space, but continuity of the map ϕ can be studied. If we add to the convergence rule (1), the requirement of convergence of the derivatives,

$$(2^*) \sup_{x \in X} \| Dp(x) - Dp^j(x) \| \rightarrow 0,$$

then P becomes a complete topological space with the composite of the two convergence rules (1*) and (2*): this is because the space of C^1 vector fields with the C^1 sup norm is closed. The results given here are valid in both topologies.

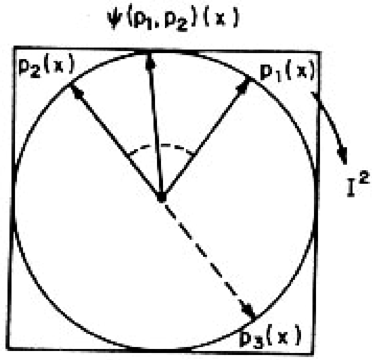

FIGURE II
The case of two voters and two-dimensional choices: the choice space is the unit cube I^2. The space of linear preference can be identified with the circle S^1.

THEOREM 1. There is no aggregation rule $\phi: P^k \to P$ that satisfies

(1) ϕ continuous;

(2) ϕ is anonymous (i.e., $\phi(p_1, p_2, \ldots, p_k) = \phi(p_{\Pi_1}, \ldots, p_{\Pi_k})$, where $\Pi_1, \ldots, \Pi_n$ denotes any permutation of the indices $1, \ldots, k$)[3] and

(3) ϕ respects unanimity (i.e., $\phi(p, \ldots, p) = p$ for all $p \in P$).

This is also true when all individual preferences are additive, or even linear, i.e., having constant gradient.

Proof. We shall first discuss a particular case in order to give a geometrical argument underlying the general result. Let the choice space X be two-dimensional, for example, a unit cube denoted I^2 as in Figure II. Assume, in addition, that the space of preferences consists of linear preferences only. Denote this space Q.

Let 0 be the center of X. Then each preference in Q can be uniquely identified in this particular case by one vector p of unit length, the normal to all the indifference surfaces of the preferences, i.e., by a point in the circle S^1. In this very special case and with two individuals, the theorem states that there exists no continuous map ψ assigning to each couple of points (p_1, p_2) in S^1 a third point p in S^1, such that

$$\psi(p_1, p_2) = \psi(p_2, p_1)$$

3. A rule ϕ is *anonymous* when the value of ϕ is invariant under permutation of its arguments; i.e., the order of the voters does not affect the results. Note that anonymity is not an appropriate concept when the elements of the choice space X are allocations of private goods. Further arguments must be given for this case, since interchanging preferences in this case should entail a corresponding interchange of consumption bundles.

(i.e., the value of ψ does not depend on the order of the preferences p_1 and p_2), and such that if $p_1 = p_2$, then

$$\psi(p_1,p_2) = p_1(=p_2).$$

We shall now discuss why two seemingly obvious candidates for such a map ψ actually fail to satisfy the conditions.

Let p_1 and p_2 be two vectors in S^1 and let $\psi(p_1,p_2)$ be the unit vector in a direction that is determined by half the angular distance between the two vectors p_1 and p_2 (see Figure II).

Now let p_1 rotate around the boundary of the circle S^1 clockwise (starting from p_1). Then as $p_1 \to p_2$ (completing the rotation of the circle), by definition of ψ, $\psi(p_1,p_2)$ must converge to p_3. However, by condition (3) $\psi(p_1,p_2)$ must also converge to p_2, by continuity, and since p_1 converges to p_2. Therefore, this rule does not satisfy the conditions. A similar problem would arise if the rule assigns half the larger angular distance between p_1 and p_2.

Another plausible candidate rule of aggregation would be one induced by the vector addition in R^2. However, the rule $\phi(p_1,p_2) = p_1 + p_2$ does not define an ordinal preference, since even if p_1 and p_2 are of unit length, $p_1 + p_2$ may have any length from 0 to 2. In order to obtain an ordinal preference, the addition rule could be modified by the normalization,

$$\phi(p_1,p_2) = \frac{\phi p_1 + p_2}{|| p_1 + p_2||}.$$

However, this newly proposed rule presents several difficulties. One is that, since $p_1 + p_2 = 0$ when $p_1 = -p_2$, the outcome of $\phi(p_1, -p_1)$ is not well defined, since the denominator in the definition is zero. Therefore, the alternative for the rules induced by vector addition is either to violate the condition that social preferences be ordinal (which goes against the spirit of much of the social choice literature because of the usual problems associated with utilitarian analysis), or else to accept a social preference that is left undefined in a number of cases.

If one attempts to accept the latter, extending the rule so that it assigns the outcome

$$\frac{p_1 + p_2}{|| p_1 + p_2 ||},$$

whenever $p_1 \neq -p_2$, and it assigns the vector zero as an outcome when $p = -p_2$, this violates the axiom of continuity of the social rule, as can be checked by examining the sequence $|\phi(p_1,p_n)|$, as $p_n \to -p_1$.

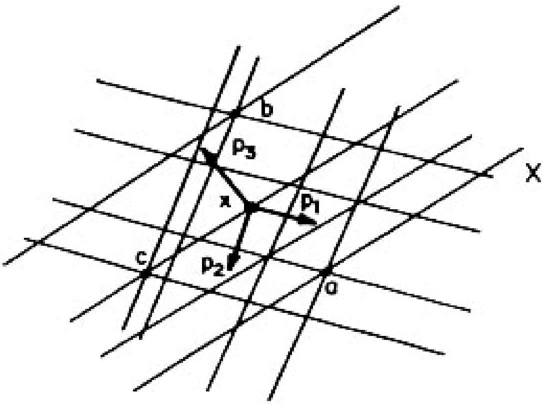

FIGURE III
The Condorcet triple is represented here by linear indifference surfaces of three preferences p_1, p_2, and p_3. The three arrows indicate the most desirable directions of p_1, p_2, and p_3. The three arrows indicate the most desirable directions of p_1, p_2 and p_3 at the choice x in X. a, b, and c are choices in X: p_1 prefers a, b, c (in that order), p_2 prefers c, a, b, and p_3 prefers b, c, a.

One may then attempt to accept some violation of the axioms, e.g., that ϕ be discontinuous in certain profiles, by arbitrarily assigning $\phi(p_1, -p_1) = 0$ when $p_1 = -p_2$. As shown in Section IV, this procedure will still cause difficulties.

It should be noted that in this case (preferences in Q) one is not only accepting the existence of discontinuities, but also the possibility that the social preference be *indifferent among all choices*. This is because when $\phi(p_1,p_2) = 0$, the outcome, being the zero vector in R^2, represents a linear preference that gives identical value to all choices (its indifference surface is all of R^2), *not* a preference with just one (or several) of its gradient vectors equal to zero. This represents a significant form of indeterminacy of the social rule.

The problem posed by this proposed rule becomes even more evident as we consider the case of three or more voters, as is done in most of the social choice literature. For instance, with three voters, $\phi(p_1,p_2,p_3)$ will be zero when $p_1 = -(p_2 + p_3)$. However, it can easily be checked that these three preferences p_1, p_2, and p_3 in fact define a Condorcet triple for certain choices a, b, and c in X. For example, if a, b, and c are as in Figure III, there p_1 prefers a, b, and c in that order, p_2 prefers c, a, b, *and* p_3, b, c, a, respectively. Therefore, the preferences p_1, p_2, and p_3, and the choices a, b, c define a Condorcet triple. Since $\phi(p_1,p_2,p_3) = 0$, this implies that the rule assigns to this Condorcet triple a social preference that is indifferent over the three choices, a, b, and c, and, moreover, among all other possible choices in X. Evidently, this cannot be considered a satisfactory solution of the social choice problem; in particular, if such trivial outcomes were acceptable, neither Condorcet's nor Arrow's paradoxes would hold. It is easy to check from Figure III that, for the addition rule described above, this problem will arise not just on one set of three choices (a,b,c) but also on a set of positive measure of choices (a',b',c') in R^2.[4]

A similar problem arises with an arbitrary number of voters $k (k \geq 3)$ as one can always consider a subset of three voters that form a Condorcet triple.

We shall now discuss a line of proof for the whole space of preference P. Note that even though for the restricted domain of linear preferences Q (i.e., when all individual preferences are in Q) the result holds, it is of interest to prove the result for the case where the social preferences are in a larger space P. This is because the lack of existence of an aggregation rule for linear preferences yielding a linear aggregate preference does not imply the lack of existence of a rule of aggregation that yields a *nonlinear* social preference.

Consider now the choice space X to be of dimension $n + 1$, $n \geq 1$. If x is a given choice in X, then each preference p in P determines uniquely a point z in the n dimensional sphere S^n of radius one, given by the intersection of the preferred direction vector $p(x)$ with S^n (see Figure IV for $n = 3$). This construction determines a map Π from the space of preferences P to the sphere S^n which is continuous. The image of this map covers S^n, because to any vector v in S^n one can assign (in a continuous manner) a preference with gradient v at x. Furthermore, as shown in Chichilnisky [1980], if a continuous aggregation map ϕ existed with the properties (2) and (3), then this

4. It can be seen that allowing the social choice rule to assign a trivial outcome (i.e., the total indifference) to a profile of linear individual preference in P, is tantamount to allowing the social choice rule to be a *set valued* function (i.e., a correspondence) rather than a proper function. This means that the rule assigns a *set* of social preferences to one profile, rather than a unique preference. The optimal choices of a set of preferences are then given by the set of optimal choices of each preference. For example, if $\phi(p_1,p_2,p_3) = 0$ when $p_1 = -(p_2 + p_3)$, the set of corresponding optimal social choices is the whole choice space. It is well-known that this produces a drastic difference in some of the better known results in social choice theory. For instance, neither Arrow's impossibility theorem, nor the Gibbard-Satterthwaite results on manipulation (that derive from Arrow's) hold in general when the social rule is a set valued function. In an extreme case, this would allow the assignment (in Arrow's theorem) of the total indifference to the Condorcet triple—thus destroying his proof. In our framework, as seen above, the Condorcet triple appears for three preferences p_1,p_2,p_3 when $p_1 = -(p_2 + p_3)$ if ϕ is derived from an addition rule. We saw that our axiom of continuity is violated when $\phi(p_1,p_2,p_3) = 0$. Therefore, it appears that the continuity axiom is violated precisely in the typical cases in which the axioms of Arrow's theorem become mutually inconsistent, i.e., on Condorcet triples.

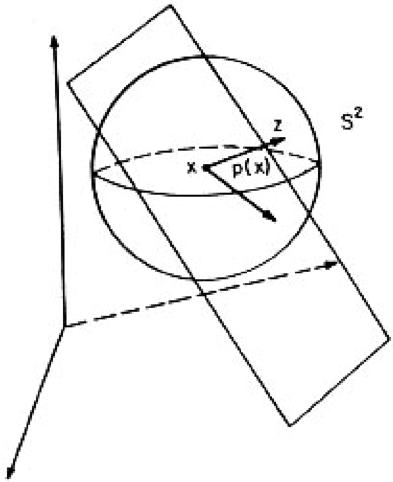

FIGURE IV

would imply existence of another continuous map ψ from the product of the S^n sphere k times with itself to S^n, i.e.,

$$\psi:(S^n)^k \rightarrow S^n$$

that satisfies the same conditions (2) and (3), on its domain of definition. ψ is defined by studying the effect of the map ϕ on the sphere S^n, when combining ϕ with the map II described above. As shown in Chichilnisky [1980], however, such a map ϕ cannot exist.

In order to give a geometrical proof of this fact, we shall again focus on a particular case, where there are 2 individuals and 2 goods (i.e., $k = 2$ and $n + 1 = 2$), and preferences are linear. The general proof for any number of individuals and any number of commodities requires the use of algebraic topology, and is given in Chichilnisky [1980]. The map ψ can then be represented as a map from $S^1 \times S^1$ into S^1. $S^1 \times S^1$ can be represented in R^3 as a two-dimensional torus T^2, or, equivalently, as the product space $S^1 \times S^1$, as in Figure V below.

The respect of unanimity condition implies that the map ψ restricted to the diagonal D in Figure V is the identity, i.e., $\psi(p,p) = p$.

On the other hand, by respect of anonymity, $\psi(q,p) = \psi(p,q)$, for all p and q in S^1. We shall show that these two properties of the map ψ are contradictory when ψ is continuous. A topological argument will be used; in this special case the argument can be simply stated in geometrical terms.

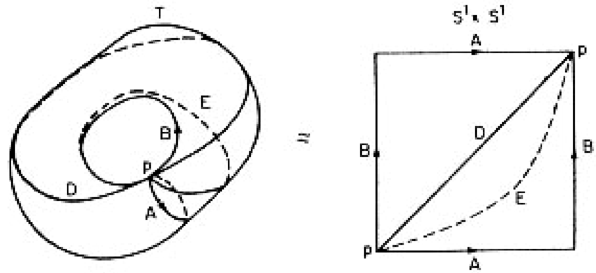

FIGURE V
The torus T is equivalent to the right-hand side figure $S^1 \times S^1$ when the two sides A are identified and the two sides B are identified as well, as indicated by the arrows.

The *degree* of a continuous map $f: S^1 \to S^1$ is now defined as the number of times $f(S^1)$ wraps completely around its image S^1. For instance, if f is the identity map, degree $(f) = 1$; if $f(p) = 2p$ (in radians), degree $f = 2$; if $f(S^1)$ does not cover S^1, degree $(f) = 0$.

Since the set D is equivalent to a circle S^1, we can define the degree of ψ when restricted on D, since $\psi/_D{:}D \to S^1$. Since $\psi(p,p) = p$,

$$\text{degree } \psi/_D = 1. \tag{1}$$

We can also define the degree of ψ restricted to the sets $A = (p \times S^1)$ and $B = (S^1 \times p)$, since both are also equivalent to circles. The respect of anonymity condition then will imply that

$$\text{degree } \psi/_A = \text{degree } \psi/_B. \tag{2}$$

Since the circle D can be continuously deformed into the union of the circles A and B (see Figure V), it follows that the degree of ψ on D should equal the degree of ψ on the union $A \cup B$; this is also discussed in Chichilnisky [1979]. We thus have

$$\text{degree } \psi/_D = \text{degree } \psi/_{A \cup B}. \tag{3}$$

On the other hand, the degree of ψ on the union $A \cup B$ is the sum of the degrees of ψ on A and on B, i.e.,

$$\text{degree } \psi/_{A \cup B} = \text{degree } \psi/_A + \text{degree } \psi/_B. \tag{4}$$

From (1), (2), (3), and (4), we obtain a contradiction, since these equations imply that one is an even number. Therefore, such a continuous rule ψ satisfying anonymity and respect of unanimity cannot exist. This completes the proof of this special case.

III. Relationship with Arrow's Impossibility Theorem

Two of the conditions in Theorem 1 are obviously related to Arrow's axioms: respect of unanimity and anonymity. While respect of unanimity is weaker than his Pareto property (which requires that if all individuals prefer alternative a to b, so does the aggregate), and anonymity is stronger than his nondictatorship condition, it has been shown in Chichilnisky [1982] that a problem of aggregation similar to that of Theorem 1 arises: the Pareto condition is topologically equivalent to the existence of a dictator.

Our requirement of continuity is therefore the main difference from his analysis. He postulates, instead, the axiom of independence of irrelevant alternatives. This is the most controversial of his axioms, and it seems worth discussing its relation with continuity.

In a first analysis it can be seen that the condition of continuity of the social aggregation rule ϕ is unrelated to the axiom of independence of irrelevant alternatives, i.e., neither implies the other. Further, it is clear that the continuity condition may be in contradiction with the axiom of irrelevant alternatives. This is discussed also by McManus [1977], who provides examples restricted to the domain of increasing preferences. For instance, continuity implies some relationship, even if very weak, among the social preference at one alternative x and the social preference at a sequence of alternatives x^n approaching x. However, such relationships should be ruled out by the axiom of independence of irrelevant alternatives. On the other hand, certain rules (such as dictatorial rules) will satisfy both continuity and the independence axiom.

Continuity is a natural assumption that is made throughout the body of economic theory, and it is certainly desirable as it permits approximation of social preferences on the basis of a sample of individual preferences, and makes mistakes in identifying preferences less crucial. These are relevant considerations in a world of imperfect information. However, continuity can be considered to be a demanding condition, since it applies to the limiting behavior of sequences of preferences. It can also be interpreted as a global independence axiom, since only when the preferences are "infinitesimally close" should the aggregation rule assign them related outcomes. If two profiles of preferences are quite distinct, the outcomes assigned to each by a continuous social rule need bear no relation.

Finally, we shall discuss a seemingly substantial difference between Arrow's results and ours that, on closer analysis, proves not to be so. There exist restricted domains of preferences (such as convex and increasing preferences) in which Arrow's impossibility theorem

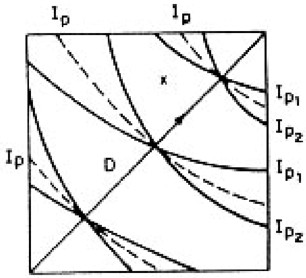

FIGURE VI
The curves I_{p1} and I_{p2} indicate indifference surfaces for the preferences p_1 and p_2, respectively. The dotted indifference curves correspond to the averaging of the indifference curves of p_1 and p_2, i.e., to the preference p.

holds; yet these domains do admit continuous aggregation rules that are anonymous and respect unanimity. For example, when individual preferences are all convex and increasing, the averaging rule that assigns to each couple of indifference surfaces corresponding to the different individuals, say I_{p1} and I_{p2}, the average of the two, say I_p in Figure VI, satisfies conditions (1), (2), and (3) in Theorem 1. More general examples of this type can also be constructed for such restricted domains of preferences (see Chichilnisky [1976]).

However, it can be argued that the above case, where everybody agrees everywhere on the existence of one direction (given by the arrow in Figure VI), along which everybody's utility improves, does not pose a typical problem of social choice. One expects that conflict may arise when there are opposite interests at work for at least *some* alternatives within the choice space, e.g., when two agents do not have everywhere monotonic preferences, increasing in the same directions. Such examples where optimal directions are opposed, appear typically in the Edgeworth-box diagrams. Other cases where monotonicity cannot be expected are when the choice space is the set of all feasible allocations of an economy, for instance, a set $X = \{(x_1,x_2,x_3): \Sigma_i x_i \leq x_0\}$, where the vectors x_1, x_2, x_3 in R^n denote the consumption allocation of n goods among three consumers, and the vector x_0 indicates total fixed resources of the n commodities. This is a typical case in which even though individual preferences over unrestricted consumption vectors in R^{n+} may be monotonically increasing, individual preferences over the feasible space X will in general increase in any possible direction at each choice in X.

However, a closer analysis of the role of the axiom of independence of irrelevant alternatives proves that conflicts of this sort are also necessary in the proof of Arrow's paradox, even though his paradox holds on domains of monotonic preferences.

The important point is that in view of his axiom of independence, Arrow's problem can be restricted to the study of preferences over three choices. Since his paradox requires the existence of a Condorcet triple or, in the case of two voters only, the existence of a similar form of circularity of preferences,[5] the domain needed to obtain Arrow's paradox does not satisfy what one usually understands by monotonicity of preferences. This can be seen as follows: since Arrow assumes independence of irrelevant alternatives, the problem of aggregation can be reduced to the study of three choices, say a, b, and c and three preferences, p_1, p_2, and p_3, which form a Condorcet triple. The Condorcet triple is usually described as a set of three preferences p_1, p_2, and p_3 over three choices, a, b, and c, such that p_1 prefers a, b, c (in that order); p_2 prefers c, a, b; and p_3 prefers b, c, a (see Figure III). It is difficult to think of these three preferences as all having one preferred direction or a common monotonicity property. Most intuitively reasonable definitions of monotonicity of preferences when restricted to three choices would rule out such Condorcet triples or circularity of preferences, and thus Arrow's paradox does not hold either for domains of preferences where such types of monotonicity are required.

IV. Problems of Aggregation with Vanishing Gradients

In Section II we considered individual and social preferences with continuously defined (normalized) gradients in the interior of the choice space X. In this section we shall consider a larger space of individual and social preferences, denoted by R, which also contains preferences whose gradients may vanish in interior points of the choice space X. A restriction we impose on preferences in R that for each $p \in R$ the set of choices in X, where the gradient $p(x)$ vanishes, be either a set of measure zero in X or the whole of X; i.e., we assume that for each preference $p \in R$, either $p(x) \equiv 0$ for all x in X, or else there exists a set S of measure zero of choices on which $p(x)$ is zero. We endow R with the topology of pointwise convergence of gradients

5. In Arrow [1950] the paradox was also proved for two individuals and three alternatives (see Section II, p. 340). However, even if properly speaking, two voters cannot form a Condorcet triple, a similar form of circularity appears that rules out any usual monotonicity assumption. For example, on p. 341, Arrow assumes individual 1 prefers x to y and y to z, while individual 2 prefers y to z and z to x. Later on, it is assumed that a voter exists who prefers z to x to y in that order. Therefore, the domain of preferences contains those represented in the diagram of Figure III by gradients. The same reasoning that implies that Condorcet triples are inconsistent with monotonicity of preferences applies here.

outside of the set of choices where these gradients vanish. The rule of convergence is thus

$$\{p^j\} \to p \text{ iff } \sup_{x \in X - \bigcup_{j=1}^{\infty} S_j} \| p^j(x) - p(x) \| \to 0,$$

where $S_j \subsetneqq X$ is the set of choices in which $p^j(x)$ vanishes.[6] Thus, in particular, $p^j(x) \to p(x)$ for all $x \in X$, except perhaps on a subset of measure zero of X.

We now require the aggregation rule ϕ to satisfy a *Pareto condition:* if x is preferred (or indifferent) to y for all preferences $p_1, \ldots, p_k$, and $\phi(p_1, \ldots, p_k) = p$, then x is preferred (or indifferent) to y according to p.

THEOREM 2. There is no social aggregation rule $\phi: R^k \to R$ such that
(1) ϕ is continuous;
(2) ϕ is anonymous;
(3) ϕ satisfies the Pareto condition.
This is also true if all individual preferences are linear.

Proof. We assume that there are $n + 1$ commodities, and k voters, $n \geq 1$, $k \geq 2$.

Consider the subset $\overline{Q}$ of R consisting of all linear preferences. Each preference in $\overline{Q}$ can therefore be identified with a point in the set,

$$S^n \cup \{0\},$$

where $\{0\}$ indicates the zero vector in R^{n+1}. Obviously, if a rule $\phi: R^k \to R$ existed satisfying (1), (2), and (3), it would induce a continuous map,

$$\overline{\phi}: \overline{Q}^k \to R,$$

satisfying anonymity and the Pareto condition, $\overline{\phi}$ being the restriction of ϕ to $\overline{Q}^k$.

Now, for any given choice x in X, the preferences in R define a set of gradients at x,

$$R_x = \{p(x): p \in R\},$$

which is in a one-to-one correspondence with the set $S^n \cup \{0\}$. It follows that for each $x \in X$, $\overline{\phi}$ induces a map,

$$\psi: \overline{Q}^k \to R_x;$$

6. With this topology R is not a complete topological space, but the continuity of the aggregation rule ϕ is still well defined.

i.e., $\psi:(S^n \cup \{0\})^k \rightarrow S^n \cup \{0\}$. The definition of the topology on R implies that ψ restricted to $(S^n)^k$ is continuous with the usual product topology on $(S^n)^k$.

Note that ψ must respect anonymity on $(S^n \cup \{0\})^k$. Also, since the Pareto condition applies for any choice y as close to x as desired, it follows that for any $(p_1, \ldots, p_k) \in (S^n \cup \{0\})^k$, the outcome $\psi(p_1, \ldots, p_k)$ must be in the cone generated by $(p_1, \ldots, p_k)$ in R^{n+1} (also see Chichilnisky [1982]). In particular, if $n = 1$ and $k = 2$, then the outcome

$$\psi(p_1,p_2)$$

must belong to the circular segment of S^1 between p_1 and p_2.

Now, $S^n \cup \{0\}$ has two connected components,[7] and $(S^n \cup \{0\})^k$ has 2^k. Consider the connected component of $(S^n \cup \{0\})^k$, consisting of profiles of nonvanishing vectors in $(S^n \cup \{0\})^k$, i.e., $(S^n)^k$. Since the image under a continuous function of a connected set is connected, and ψ restricted to $(S^n)^k$ is continuous by definition of the topology on R, then either $\psi(S^n)^k \subset S^n$ or $\psi(S^n)^k \subset \{0\}$. Since ψ satisfies the Pareto condition, in particular, ψ respects unanimity. It follows that $\psi(S^n)^k \not\subset \{0\}$, implying that $\psi(S^n)^k \subset S^n$.

Therefore, if ϕ exists satisfying (1), (2), and (3), it would induce a continuous map,

$$\psi:(S^n)^k \rightarrow S^n,$$

the restriction of ψ to $(S^n)^k$, which is anonymous and respects unanimity, since ψ is Pareto. Such a map cannot exist by Theorem 1 of Chichilnisky [1980]. Therefore, ϕ cannot exist.

We shall now give a geometric argument proving this result for the special case of two voters and two-dimensional commodity spaces. As above, let $\overline{Q}$ denote the space of linear preferences in R; here $\overline{Q} = S^1 \cup \{0\}$.

The arguments of above show that if $\phi:R^2 \rightarrow R$ existed satisfying (1), (2), and (3), then for each x_0 in X there would exist a continuous map,

$$\psi:S^1 \times S^1 \rightarrow S^1,$$

which is anonymous and Pareto, where the image S^1 denotes gradients at x_0. Let p be a given preference in S^1, and let $\{p^j\}$ be a sequence in S^1. Then as seen above $\psi(p,p^j)$ must belong to the circular segment

7. A topological space X is connected if it cannot be expressed as the union $A \cup B$, where A and B are both open and closed sets with empty intersections. A connected *component* of a disconnected set X is a subset of X that is connected and is not strictly contained in any connected subset of X.

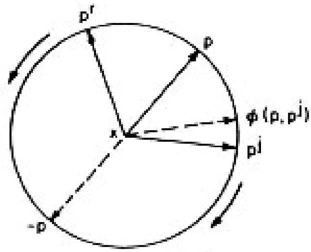

FIGURE VII

of S^1 between p and p^j, by the Pareto condition (see Figure VII). However, as $\{p^j\} \to -p$ from the right, $\psi(p,p^j)$ must belong to the closed (right-hand) segment of circle $(p,-p)$, and as $\{p^j\} \to -p$ from the left, $\psi(p,p^j)$ must be in the closed (left-hand) segment $(-p,p)$. Therefore, by continuity of ψ, only three mutually exclusive possibilities arise for the aggregate of p and $-p$: either $\psi(p,-p)$ equals p, or $\psi(p,-p)$ equals $-p$, or $\psi(p,-p)$ is 0 at x; i.e., the gradient of the social preference vanishes at x. Assume first that $\psi(p,-p) = p$.

Consider the restriction of the map ψ on the circle $p \times S^1$, as in Theorem 1 (see Figure VII). Since the image of $p \times S^1$ under ψ does not cover S^1 (the value $-p$ is never assumed), it follows by definition that the degree of of ψ restricted to $A = \{p \times S^1\}$ is zero. By the condition of anonymity, the degree of ψ restricted to $B = \{S^1 \times p\}$ is also zero. Therefore, the degree of ψ on $A \cup B$ is zero. However, since Pareto implies respect of unanimity, the degree of ψ on the diagonal D is one, as seen in Theorem 1. This implies a contradiction because the degree of ψ on D must equal the sum of the degrees of ψ on A and on B, as in Section II. A similar argument can be given to show that the case $\psi(p,-p) = -p$ is equally contradictory with the conditions. Therefore,

$$\psi(p,-p) = 0 \qquad \text{at } x_0.$$

However, by continuity of ψ, this implies that $\psi(S^1 \times S^1) \subset \{0\}$, contradicting the fact that

$$\psi(p,p) = p \qquad \text{for all } p \epsilon S^1,$$

which follows from the Pareto condition. Therefore, ψ does not exist. Since x_0 is arbitrarily chosen in S^1, this implies that no continuous map,

$$\psi: R^2 \to R,$$

exists satisfying (2) and (3), completing the proof.

Remark. Note that the respect of unanimity condition cannot replace the Pareto condition in the proof of Theorem 2. This is because the Pareto condition on a rule $\phi: P^k \to P$ is also valid locally; i.e., it implies that for *any* choice x in X, and for *any* profile of preferences $(p_1, \dots, p_k)$ in P^k, if the gradients $p_1(x), \dots, p_k(x)$ span a cone $V(p_1(x), \dots, p_k(x))$ in R^{n+1}, then the gradient of $\phi(p_1, \dots, p_k)$ at x, $\phi(p_1, \dots, p_k)(x)$ must be in $V(p_1(x), \dots, p_k(x))$. The proof of the theorem uses this fact on profiles of linear preferences,

$$(p_1, \dots, p_k) \in (\overline{Q})^k \subset R^k.$$

The respect of unanimity condition, instead, puts no constraint on the gradients locally; in particular, if ϕ respects unanimity but is not Pareto, the gradient $\phi(p_1, \dots, p_k)(x)$ may bear no relation to the gradients $p_1(x), \dots, p_k(x)$ when for some i,j and some $y \in X$

$$p_j(y) \neq p_i(x).$$

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