# COMPARATIVE STATICS OF LESS NEOCLASSICAL AGENTS

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# COMPARATIVE STATICS OF LESS NEOCLASSICAL AGENTS*

BY GRACIELA CHICHILNISKY AND PETER J. KALMAN†[1]

## 1. INTRODUCTION

In recent years demand and producer theories have been extended to models where the economic agents exhibit more complex characteristics than those of the neoclassical agent.[2] The optimization problem of these less neoclassical agents includes cases where the objective functions depend also on parameters; there are many (not necessarily linear) constraints, and non-convexities. For example, agents' preferences among commodity bundles may be parameterized or influenced by prices as in Veblen and Scitovsky models [6, 3], or real balances may enter the utility functions [9]. Other models where the objective functions are parameterized are those of choice under uncertainty and with imperfect information. Nonconvexities on the side of the constraints (technology) are naturally induced by informational variables; in models with uncertainty as many constraints may appear as states of nature.

A natural question concerning the models discussed above is to what extent do the comparative statics results of the neoclassical theory still apply. In particular, since it is known that the Slutsky matrix and its properties of symmetry and negative semi-definiteness are not preserved in general [6], one can, at most, hope to obtain conditions on the classes of models (classes of objective functions and constraint functions) in which these properties are still satisfied.[3]

Even though by nature comparative static properties are essentially local, the techniques involved so far in their proofs mostly used arguments requiring convexity assumptions of the objective and constraint functions. Since comparative static theorems concern the signs of partial derivatives in some neighborhood of an equilibrium point, these global assumptions place more stringent restrictions on the objective and constraint functions than are necessary.[4]

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† *Editor's note*: We regret to report the untimely death of Peter Kalman.

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[2] For example, in neoclassical consumer theory the objective function (utility) is usually assumed to be concave, the constraint (budget) linear, and no parameters affect the utilities. In producer models, there is usually only one constraint, and convexity assumptions are in general made.

[3] These properties have been recovered for certain separable classes of objective and constraint functions in some of these more general types of models, mostly under convexity assumptions and with special restrictions for each particular case [7].

[4] Since 1970 there has been an upsurge in the study of local properties of equilibria starting

(*Continued on next page*)

Theorems 1 and 2 contain local results about solutions to a general class of constrained maximization models; in Theorem 1 we study generic differentiability of the solutions and in Theorem 2 we give a generalized Slutsky type decomposition. The techniques used in Theorem 1 to obtain generic differentiability of the solutions are related to and extend results of Debreu [5], who parameterizes the agents by their endowments in a general equilibrium model, and Smale [12]. We extend those results by considering as paramenters both the objectives and the constraint functions. However, even though the parameters of the objectives and constraints include elements of infinite dimensional function spaces, here we do not use Thom's trans versality theorem as in [12]. The genericity in these parameters is obtained here by proving a new transversality theorem, which uses Sard's theorem [1], and with respect to a topology described by the proximity of the values of the functions and their derivatives, which seems natural for spaces of economic agents. The results which are valid for compact subspaces of the commodity space, admit an extension to noncompact commodity spaces if one uses the Whitney topology as, for example, in [12] or [8]. The derivation of the generalized Slutsky operator of Theorem 2 becomes more complicated here than in the usual models because of the many constraints, and the operator obtained is of a slightly different nature. One reason is that compensation can be performed in different manners here, since there are many constraints. Also, the existence of parameters induces new effects that do not exist in nonparameterized models, and the classical properties of symmetry and negative semi-definiteness are not, in general, preserved [6]. Finally, we consider in this paper, for the case of models with price dependent preferences, those preferences where the objects of choice are "quantity-price situations" also called unconditional preferences. An alternative way of looking at price dependent preferences is where the objects of choice are only quantities, for a fixed set of prices, also called conditional preferences; for a discussion, see, for instance, [3] and [10]. The "compensated" demand functions, however, are only defined in the case of unconditional preferences [10]: our results apply to cases of households and firms with unconditional preferences. Propositions 1 and 2 give sufficient conditions for recovering symmetry and negative semidefiniteness properties in our general framework. Results related to this paper

## 2. RESULTS

We first prove generic results on local uniqueness, differentiability and Slutsky type decompositions of optimal solutions to constrained optimization problems

*(Continued)*
with the leading article by G. Debreu who introduced tools of differential topology to study, among others, problems of existence, local uniqueness and stability of equilibria [5]. Independently, interest in local properties arose from models where there may be many equilibria positions, for instance, when the utilities cost or production functions are not necessarily conves (concave).

are contained in [4].
with parameters entering the objective function and constraints — which can be linear or nonlinear. This formulation contains the models discussed above, and also the neoclassical producer and consumer models. Convexity assumptions on either the objective functions or the constraints are not required; the special cases where the objective function is concave and increasing and the constraints are convex yield optimal functions — as opposed to correspondences — with above properties.

The problem studied here is that of an agent maximizing a constrained objective:

$$\text{(P)} \qquad \max_{x\in X} f(x, a) \qquad \text{subject to} \quad g(x, a) = b$$

where $X$ is a compact subset of $R^{n+}$[5] whose interior is diffeomorphic to a ball in $R^n$, $a \in A$, $b \in B$, $A$ and $B$ are similar type subsets of $R^{m+}$ and $R^{l+}$, respectively, and $n > l$. An agent is characterized by an objective function $f$ and by a constraint $g$. Therefore, the space of all possible agents can be identified with the product of the space of admissible objectives and constraints. Let the space of objective functions denoted $D$ be $C^k(X \times A, R^+)$, the space of maps from $X \times A$ to $R^+$ which are increasing in $x$ and $k$-times continuously differentiable in a neighborhood of $X \times A$, and let the space of constraints denoted $E$ be $C^k(X \times A, B)$, where $k \geq 2$.

We now briefly discuss the topology of the function spaces we consider. Let $Y$ be a compact ball. Then the space $C^k(Y, R)$ can be given the $C^k$ norm topology defined by:

$$\|f\|_k = \sup_{y\in Y} \{|f(y)|, |D^1 f(y)|, \dots, |D^k f(y)|\}$$

where $D^i f$ denotes the $i$-th derivative of $f$. Let $\bar{D}$ denote a $C^1$ bounded subset of $D$.[6] We shall also consider here the special cases of increasing concave objective functions and convex constraints: Let $D_0$ be the space of $C^k$ functions $f$ defined on a neighborhood of $X \times A$ with values in $R^+$ which are increasing and concave on the variable $x \in X$, and let $E_0 \subset E$ be the subset of functions of $C^k(X \times A, B)$ which are convex on $x \in X$. Let $\bar{D}_0$ denote $\bar{D} \cap D_0$.

In the next result we study properties of the optimal solutions to problem (P), denoted $h_{f,g}(a, b)$. Note that $h_{f,g}(a, b)$ is, in general, a correspondence.[7] A solution is called interior if it is contained in the interior of the set $X$.

THEOREM 1. *For an open and dense set of objective functions $f$ in $\bar{D}$, and*

[5] $R^{n+}$ denotes the positive orthant of $R^n$.

[6] This assumption on $\bar{D}$, which does not imply compactness of $\bar{D}$, could be weakened by the use of different topologies on $D$, such as those of [12, 8].

[7] In the classical consumer case $h_{f,g}(a, b)$ represents the demand vector, $x$ a commodity bundle, $b \in R^+$ income, and $a$ the price vector. Also, $g(x, a) = x \cdot a = b$ represents the budget constraint and $f(x, a) = u(x)$ the utility function.

*any given constraint $g$ in $E$, the interior solutions of problem (P) above define locally unique $C^1$ functions $h_{f,g}(a, b)$ on a subset of $A \times B$ which contains an open and dense set. This is also true for the globally defined functions $h_{f,g}$ $(a, b)$ for $f$ in $\bar{D}_0$ and $g$ in $E_0$.*

PROOF: For any $g$ in $E$, let

$$\psi\colon \bar{D} \times A \times B \longrightarrow C^{k-1}(X \times R^l, R^n \times B)$$

be defined by

$$\psi(f, a, b)\,(x, \lambda) = \left(\frac{\partial}{\partial x} f + \lambda \frac{\partial}{\partial x}\, g,\, g - b\right)$$

where $\lambda \in R^l$.

We first note that for each $a$, $b$ in $A \times B$, $\psi(\cdot, a, b)$ is continuous as a function on $\bar{D}$ since the map

$$\partial\colon C^k(X, R) \longrightarrow C^{k-1}(X, R^n)$$

defined by

$$f \longrightarrow \frac{\partial}{\partial x} f$$

is continuous in the respective $C^k$ and $C^{k-1}$ topologies. Thus, $\psi$ is itself a continuous map.

We now consider the restriction of $\psi(f, a, b)$ on $X \times B_0$, where $B_0$ is a compact ball of $R^l$ which contains the $\lambda$'s in the kernel of $\psi(f, a, b)(x, \cdot)$ for $x \in X$.[8] For simplicity, denote $\psi(f, a, b)|_{X \times B_0}$ by $\psi(f, a, b)$.

Let $B_1 = X \times B_0$. Thus,

$$\psi(f, a, b) \in C^{k-1}(B_1, R^n \times B).$$

Let $\theta$ be the set of maps $\xi$ in $C^{k-1}(B_1, R^n \times B)$ such that $\xi \pitchfork 0$.[9] Since $B_1$ is compact, by the openness of transversality theorem (see [1]), $\theta$ is an open set.

Consider now the restriction of the $C^{k-1}$ norm topology on the subset I of $C^{k-1}(B_1, R^n \times B)$, where $I$ is the image of $\bar{D} \times A \times B$ under $\psi$. Let $\bar{\theta} = \theta \cap I$ and let $I$ inherit the relative topology, and let $\bar{\psi}$ be defined as equal to $\psi$ on the domain of $\psi$, but having $I$ as its image. Then $\bar{\theta} = \theta \cap I$ is open in the relative topology of $I$ and by continuity of $\bar{\psi}$, $\bar{\psi}^{-1}(\bar{\theta})$ is also open in $\bar{D} \times A \times B$. Note that $\bar{\psi}^{-1}(\theta)$

[8] We define $\psi(f, a, b)$ on a subset of $X \times R^l$ which includes all $x$ in $X$ and those $\lambda$'s given by the zeros of the first order conditions of $\psi(f, a, b)(x, \cdot)$ for some $x \in X$. By [11, (30)], for all $x \in X$ and for all $(f, a, b)$ in $\bar{D} \times A \times B$ the respective $\lambda$'s in the kernel of $\psi(f, a, b)(x, \cdot)$ are contained in some compact ball $B_0$ of $R^l$.

[9] Let $M$ and $N$ be $C^k$ manifolds, $f\colon M \to N$ a $C^k$ map. We say that $f$ is transversal to a point $y \in N$ denoted by $f \pitchfork y$ if whenever $y = f(x)$, i. e., $x \in f^{-1}(y)$, then $Df(x)$ is onto, where $Df(x)$ represents the derivative of the map $f$ computed at $x$, a linear map from the tangent space of $M$ at $x$ to the tangent space of $N$ at $y$. A point $x$ in $M$ is a critical point if and only if $Df(x)$ is not onto; $x$ is a regular point if it is not a critical point. $y$ is a critical value if there exists a critical point $x$ in $M$ with $y = f(x)$; $y$ is a regular value if and only if it is not a critical value (see [1]).

is contained in the set of elements in $\bar{D}\times A\times B$ such that the corresponding interior optimal solutions $h_{f,g}(a, b)$ of (P) define locally a unique $C^1$ function[10] by the implicit function theorem (since $\frac{\partial}{\partial x, \lambda}\psi(f, a, b)$ is regular at the kernel of $\psi(f, a, b)$ if and only if it is invertible). Hence, for an open set of objective functions in $\bar{D}$, and an open set of parameters in $A\times B$, the interior solutions of (P) define locally unique functions which are $C^1$.

By Sard's theorem, (see [1]) since $k\geq 1$, the set of regular values of $\bar{\psi}(f, a, b)$ is dense in $R^n\times B$. Then, for any $\varepsilon>0$, let $(q, k)\in R^n\times B$ be a regular value of the map $\bar{\psi}(f, a, b)$ with $\|q, k\|<\varepsilon$. Define $\bar{\psi}^\varepsilon$ by:

$$\bar{\psi}^\varepsilon(f, a, b)=\psi(f, a, b)-(q, k).$$

Note that $\bar{\psi}^\varepsilon(f, a, b)\pitchfork 0$ iff $(q, k)$ is a regular value of $\bar{\psi}(f, a, b)$. If $\bar{f}=f-qx$, and $\bar{b}=b-k$, then $\bar{\psi}^\varepsilon(f, a, b)=\bar{\psi}(\bar{f}, a, \bar{b})$. Since $X$ is compact, $\bar{f}$ can be taken to be arbitrarily close to $f$ in the $C^k$ norm by choosing $\varepsilon$ small enough, and similarly, $\bar{b}$ can be chosen arbitrarily close to $b$. Therefore, since 0 is a regular value of $\bar{\psi}^\varepsilon(f, a, b)$, then $(\bar{f}, a, \bar{b})\in\bar{\psi}^{-1}(\bar{\theta})$, and thus $\bar{\psi}^{-1}(\bar{\theta})$ is also dense in $\bar{D}\times A\times B$.

If $f$ is concave and $g$ convex, i.e., if $f$ is in $\bar{D}_0$ and $g$ in $\bar{E}_0=E_0\cap\bar{E}$, then the above results also apply, proving in this case that the globally defined optimal functions of the agents are $C^1$ on an open and dense subset of $A\times B$ and of objective functions and constraints $f$ and $g$ in $\bar{D}_0\times\bar{E}_0$. This completes the proof.

*Remarks* 1. Note that the results of Theorem 1 are restricted to interior solutions of problem (P); solutions to (P) always exist by compactness of $X$. If the objective function $f$ is required to have all its hypersurfaces (or indifference surfaces) contained in the interior of $X$, then it would follow that all solutions to (P) are interior. However, since $X$ is compact, this would imply some satiation of the maximizing agent. When the choice space is $R^{n+}$ a standard assumption is to require that the indifference surfaces be contained in the interior of $R^{n+}$, in which case all solutions are interior. This boundary condition does not imply satiation since $R^{n+}$ is not compact.[11]

2. Let $\bar{E}$ be a $C^1$ bounded subset of $E$. Then if the map $\psi$ of Theorem 1 is defined instead as:

$$\psi\colon \bar{D}\times\bar{E}\times A\times B\longrightarrow C^{k-1}(X\times R^l, R^n\times B)$$

by

$$\psi(f, a, b)\,(x, \lambda)=\left(\frac{\partial}{\partial x}f+\lambda\frac{\partial}{\partial x}\,g,\, g-b\right)$$

[10] There might be elements in $\bar{D}\times A\times B$ such that the corresponding $h_{f,g}(a, b)$ defines a $C^1$ function, but are not contained in $\bar{\psi}^{-1}(\theta)$ since $(\partial/\partial x, \lambda)\psi(f, a, b)$ may be singular. Also, the boundary solutions to (P) may not be contained in $\bar{\psi}^{-1}(\theta)$.

[11] The results of Theorem I can be extended to the case where $X$ is $R^{n+}$ by the use of different topologies on spaces of $C^k$ functions, such as the Whitney topology [12, 8].

a similar proof would yield open density of the set of objective functions and constraints in which the results of Theorem 1 are true, instead of a fixed constraint $g$.

We next study existence of a Slutsky-type decomposition for the interior solutions to problem (P) above. Note that such a decomposition can only be defined in a neighborhood of $(a, b)$ if $h(a, b)$ is a $C^1$ function at $(a, b)$.

Assume $k > l$. For a given $f$ and $g$ a necessary condition for $x$ to be an interior maximum is the existence of a $\lambda$ in $R^l$ such that

$$\frac{\partial}{\partial x} f(x, a) + \lambda \frac{\partial}{\partial x} g(x, a) = 0,$$

$$\text{and } g(x, a) = b. \tag{1}$$

The following result is proven in [4].

THEOREM 2. *Let $g \in E$ be regular.*[12] *For an open and dense set of objective functions $f$ and parameters $a$ in $\bar{D} \times A$, if the corresponding Lagrangian multiplier $\lambda$ of* (1) *is strictly positive, and Z defined below exists,*[13] *then there exists locally a Slutsky-type decomposition $S(a, b)$ into linear operators, given by:*[14]

$$\begin{aligned} S(a, b) &\equiv \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} h\left(\frac{\partial}{\partial a} g\right) \\ &= \frac{\partial}{\partial a} h\Big|_{f=f_0} + \frac{\partial}{\partial b} h\left(\phi\left(\frac{\partial}{\partial a} g\right) - \mu\left(\frac{\partial}{\partial a} f\right)\right) \\ &= -\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \right. \\ &\quad \left. + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x \partial a} L\right) \end{aligned} \tag{2}$$

*where $L(x, a, \lambda, b)$ denotes $f(x, a) + \lambda(g(x, a) - b)$,*

$$Z = -\left[\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right)\right]^{-1},$$

*and $\phi$ and $\mu$ are defined as follows:*

$$\mu = \Big( 0, \ldots, 0, \underset{i\text{-th place}}{\frac{1}{\lambda_i}}, 0, \ldots, 0 \Big)$$

[12] I. e., $\frac{\partial}{\partial x} g(x, a)$ is onto for all $x$ and $a$.

[13] Existence and density of such $\lambda$'s is discussed in [2]. Generic existence of $Z$ can be proven following the techniques of Theorem 1 also.

[14] Transposes of matrices are not indicated in the statement of Theorem 2.

*and*

$$\phi = (\phi_{q,r}), \qquad q = 1,\dots, l, \quad r = 1,\dots, l,$$
$$\phi_{q,r} = 1 \qquad \text{if } \; q = r, q \neq i$$
$$\phantom{\phi_{q,r} = {}} 0 \qquad \text{if } \; q \neq r \;\text{ and }\; q \neq i$$
$$\phi_{i,r} = -\frac{\lambda_r}{\lambda_i} \qquad \text{if } \; r \neq i$$
$$\phi_{i,i} = 0; \qquad \text{if } \; b \in R^1, \phi = 0.$$

*If $f \in \bar{D}_0$ and $g \in \bar{E}_0$ the above results hold globally.*

Note that the term

$$\frac{\partial}{\partial b} h\left(\phi \frac{\partial}{\partial a} g - \mu \frac{\partial}{\partial a} f\right)$$

of the decomposition of $S(a, b)$ depends on the choice of the index $i$ through the operators $\mu$ and $\phi$. Thus the middle term of the equation (2) depends on the choice of the particular constraint which is used for the compensation (denoted by the index $i$ in the definition of the operators). However, $S(a, b)$ as given by the first and third expressions, does not depend on the choice of the index. In the cases of optimization models with only one constraint, the compensation can be done in only one way; in general, one can choose as many ways of "compensation" as there are constraints, and more: any linear combination of the constraints, for instance, could also be used as a compensating parameter, and a similar proof will follow. The basic point here is that when $df=0$ (on the surface $f=$constant), the components of the differential form $db$ are not all linearly independent. Different choices or representations of this linear dependence (of which a particular case is that given by the operators $\phi$ and $\mu$) would yield different forms of the terms of the middle expression of (2). When considering this particular expression in the middle of equation (2), which can be thought of as a "modified substitution effect", these degrees of freedom in the choice of the compensating parameter may allow one to adopt a particular way of compensation depending on the type of economic model under consideration and depending on the properties one wants to study in each particular model.

In this general framework the classical properties of symmetry and negative semi-definiteness are not, in general, preserved [6]. In Propositions 1 and 2 below, we exhibit some forms of objective functions and constraint functions for which $S(a, b)$ has these properties under certain conditions. For proofs of the next two propositions, see [4].

PROPOSITION 1. *Assume the objective function $f(x, a)$ has the form*

(i) $$f \equiv \gamma[a \cdot x] + f^1(x) + f^2(a)$$

*and the constraints $g(x, a)$ have the form*

(ii) $$g^i \equiv \delta^i[a \cdot x] + g^{i1}(x) + g^{i2}(a), \qquad i = 1, 2, \ldots, l$$

*and that all the conditions for existence of the matrix $S(a, b)$ of Section 1 are satisfied, $\gamma$, $\delta^i \in R^+$. Then $S(a, b)$ is symmetric when $m = n$ in problem (P).*

PROPOSITION 2. *Under the conditions of Proposition 1, if $f \in D_0$ and $g \in E_0$, the matrix $S(a, b)$ is negative semi-definite when $\gamma + \Sigma_i \lambda_i \delta^i \geq 0$, and where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the corresponding "Lagrangian multiplier".*

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