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Properties of Critical Points and Operators in Economics*

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Introduction

The study of the local characteristics of equilibrium positions generated by the constrained maximization of some criterion functions (such as utility, profit, cost, etc.) goes back to Antonelli's paper [2] of 1886. Certain local properties have proved to be particularly fruitful for economic theory since the early works of Slutsky, Hicks, and Samuelson [9]; they have been formulated in terms of a matrix of "compensated" terms and they concern the properties of partial derivatives of the demand function in some neighborhood of the equilibrium.

We now briefly discuss this matrix and its applications. Consider the maximization problem:

$$\max_x f(x, a), \qquad \text{subject to } g(x, a) = b, \tag{1}$$

where f is a real valued map defined on a linear space and g is vector valued, defined on a linear space. Under certain assumptions the optimal solution (also called equilibrium) vector $x = h_{f,g}(a, b)$ is a C^1 function of a and b, and, as the parameter b varies, the constraints describe a parametrized family of manifolds on which f is being maximized. In neoclassical consumer theory, for instance, f represents a utility function; x, consumption of commodities; a, prices of all commodities; and b, income. In neoclassical producer theory f represents the cost function; x, inputs; a, input prices; g, a production function; and b, an output requirement. In these models $b \in R^+$. Important results of the theory relate to the compensation operator, denoted S, given by the derivative of the optimal solution h with respect to the parameter a, restricted to the manifolds determined by the solutions of

$$f(x, a) = \text{constant};$$

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i.e., S is given by

$$(\partial/\partial a)\, h(a, b)|_{f=c}\,.$$

denoted also $S(a, b)$. Under certain assumptions, for instance, for a consumer:

$$S(a, b) = (\partial/\partial a)\, h(a, b) + h(a, b)(\partial/\partial b)\, h(a, b). \tag{2}$$

(See [9].) Equation (2) is also called the Slutsky-Hicks-Samuelson equation. In the case of the consumer, $S(a, b)$ is itself unobservable, since it represents changes in the demand due to a price change when utility is assumed constant, but the right-hand side represents two observable basic effects called the price effect and the income effect, respectively. Similar operators are found throughout the body of economic theory. The assumption that $S(a, b)$ be symmetric and negative semidefinite (SN) has theoretical and empirical implications in utility analysis.

From these two properties of a demand function h, for instance, one can, under certain regularity conditions, prove the existence of a utility function f such that the given demand h at a price income pair (a, b) maximizes f over the budget set.[1] This is also called an integrability problem (the symmetry property is related to the Frobenius property of local integrability of vector fields or preferences). In many agent models, the negative semidefiniteness property is related to problems of stability of the equilibrium [3]. These SN properties also yield most known comparative static theorems concerning maximization models in economic theory, and they have proved useful in an ample body of empirically testable results.

Even though by nature the SN properties are essentially local, the techniques involved in their proofs so far have used arguments requiring convexity assumptions of the objective and constraint functions f and g. Since comparative static theorems concern the signs of partial derivatives only in some neighborhood of an optimal solution, these global assumptions place more stringent restrictions on the objective and constraint functions than seem necessary. Local properties are of special interest in models where global uniqueness of the optimal solution may not obtain, for instance, when convexity (concavity) of the utility, cost, or production functions is not necessarily satisfied (as when there exist externalities in production or consumption [3]). Other types of economic optimization models which require further analysis are those where there are many constraints (g vector valued) and those where parameters appear in the objective function f as well as in the constraints. Examples are models of choice under uncertainty, models where consumer utilities depend on prices of the commodities as well

[1] A continuous function $h: A \times B \to X$ ($A \subset R^{n+}$ denotes a space of prices, $B \subset R^{+}$ represents a space of incomes, and $X \subset R^{n+}$ is a commodity space) is called a demand function if $a \cdot f(a, b) = b$ for all (a, b). A budget set is a set of commodity vectors in X satisfying, for a price system a in A, a budget constraint $a \cdot x \leq b$ [3].

as on the commodities themselves (Veblen and Scitovsky [3]), and monetary models [6]. These are contained in the formulation of problem (1) above. The conditions under which (2) has been proved so far do not include all of these cases; only one constraint is assumed, there is a trade off between the lack of parameters and the linearity of the constraint, and f and g are assumed to be convex (concave) throughout their domain. In certain models with parameters, the SN properties are, in general, lost [7]. A natural question is, To what extent do the results of the existent theory extend to these more general models? In this paper we study properties of the solutions of the general contrained maximization problem (1), and of a (generalized) S matrix. The results extend others in [6, 7]. We also study conditions on the functions f and g under which S has the SN properties.

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The problem studied here is (1) above; an agent maximizing a contrained objective:

$$\max_{x \in X} f(x, a) \qquad \text{subject to } g(x, a) = b,$$

where X is a compact region of the nonnegative orthant of R^n, denoted R^{n+}, with $\mathring{X}$ diffeomorphic to a ball in R^n, $a \in A$, $b \in B$, A and B similar type subsets of R^{s+} and R^{m+}, respectively, $n > m$. An agent is identified with an objective function f and a constraint g. Therefore, the space of all possible agents can be identified with the product of the space of admissible objectives and constraints. Let the space of objective functions $D = C^k(X \times A, R^+)$ be the space of maps from $X \times A$ to R^+ which are increasing in x and k-times continuously differentiable in a neighborhood of $X \times A$, and let the space of constraints be $E = C^k(X \times A, R^+)$, $k \geqslant 2$. These function spaces are given a C^k norm topology. We shall also consider here the special cases of increasing concave objective functions and convex constraints: Let D_0 be the space of increasing concave (on $x \in X$) C^k functions f defined on a neighborhood of $X \times A$ with values in R^+. Similarly, let $E_0 \subset E$ be the subset of convex functions (on $x \in X$) of $C^k(X \times A, B)$. Let $\bar{D}$ denote a C^1 bounded subset of D and $\bar{D}_0$ denote $\bar{D} \cap D_0$; $\bar{E}_0$ denotes $E_0 \cap \bar{E}$ where $\bar{E}$ is a C^1 bounded subset of E. We next study properties of the optimal solutions $h_{f,g}$, depending on the parameters (a, b). In order to define the matrix S it is necessary that $h_{f,g}(a, b)$ be a function and of class C^1.

THEOREM 1. *For an open and dense set of objective functions f in $\bar{D}$, and any constraint g in $\bar{E}$, the interior solutions of* (1) *define locally unique C^1 functions $h_{f,g}(a, b)$ on a subset of $A \times B$ which contains an open and dense set. This is also true for the globally defined $h_{f,g}(a, b)$ when f is in $\bar{D}_0$ and g in $\bar{E}_0$.*

This result is related to others by Debreu [5], given for many agent equilibria, and Smale [10]. In [5] the agents are parametrized by their endowments in R^l; here the agents are also parametrized by the utilities in the function space D. In [10] the one-agent case is parametrized also by utilities; this result improves a related one in [10] using a different technique. The idea is to study the critical points of a map ψ defined as follows. For any g in E, let

$$\psi : \bar{D} \times A \times B \to C^{k-1}(X \times B^*, X^* \times B)$$

be given by

$$\psi(f, a, b)(x, \lambda) = \left(\frac{\partial}{\partial x} f + \lambda \frac{\partial}{\partial x} g, g - b\right),$$

where $\lambda \in R^m$.

Let θ be the set of maps ξ in $C^{k-1}(B_1, X^* \times B)$ such that 0 is a regular value of ξ, denoted $\xi \pitchfork 0$ [1]. Note that $\psi^{-1}(\theta)$ is contained in the set of elements in $\bar{D} \times A \times B$ such that the corresponding interior optimal solutions $h_{f,g}(a, b)$ of (1) define locally a unique C^1 function, by the implicit function theorem (since $(\partial/\partial x, \lambda)\, \psi(f, a, b)$ is regular at the kernel of $\psi(f, a, b)$ if and only if it is invertible). There might be elements in $\bar{D} \times A \times B$ such that the corresponding $h_{f,g}(a, b)$ define a C^1 function, and are not contained in $\psi^{-1}(\theta)$ since $(\partial/\partial x, \lambda)\, \psi(f, a, b)$ may be singular. Also, the boundary solutions to (1) may not be contained in $\psi^{-1}(\theta)$. Using Sard's theorem [1] and further arguments, one shows that $\psi^{-1}(\theta)$ is an open and dense set; for a proof see [4].

Remark. If the map ψ of Theorem 1 is defined instead as

$$\psi : \bar{D} \times E \times A \times B \to C^{k-1}(X \times B^*, X^* \times B)$$

by

$$\psi(f, g, a, b)\,(x, \lambda) = \left(\frac{\partial}{\partial x} f + \lambda \frac{\partial}{\partial x} g, g - b\right),$$

then a similar proof would yield open density of the set of objective functions and constraints in which the results of Theorem 1 hold.

We now discuss a case where the commodity space X is not bounded. We need some definitions. A subset of a topological space is residual or Baire if and only if it is the countable intersection of open dense sets. By Baire's theorem a residual set in a Baire space is of the second category (and hence dense). On the space of C^k real valued functions defined on a neighborhood of R^{n+} we put the Whitney topology (see [8]). This topology is defined by giving a basis of neighborhoods $\{\mathscr{U}_h\}$ of 0: For any strictly positive continuous function $h: R^{n+} \to R$, $g \in \mathscr{U}_h$ if

$$|g(x)| < h(x), \qquad \text{for all } x \in R^{n+},$$

and

$$\| D^j(x) \| < h(x), \qquad \text{for all } X \text{ in } R^{n+}, \quad j \leqslant k.$$

The space $C^k(E^{n+} \times A, R^+)$ is a Baire space when endowed with the Whitney topology.

Corollary 1. *For each constraint g in $C^k(R^{n+} \times A, B)$, the results of Theorem 1 follow for a Baire subset of objective functions f in $C^k(R^{n+} \times A, R^+)$, and of parameters in $A \times B$.*

Proof. It follows from Theorem 1 and the properties of the topology chosen for $C^k(R^{n+} \times A, R^+)$.

We next study the S matrix. For a given f and g, a necessary condition for an interior solution of (1) is the existence of a λ in R^m such that

$$\frac{\partial}{\partial x} f(x, a) + \lambda \frac{\partial}{\partial x} g(x, a) = 0, \qquad \text{and} \qquad g(x, a) = b. \tag{3}$$

Assume $k > m$.

Theorem 2. *Let $g \in E$ be regular as a function of x. For an open and dense set of objective functions f and parameters a in $D \times A$, when the corresponding Lagrangian multiplier λ of (*) is strictly positive, and Z defined in (8) below exists,[2] there exists locally a Slutsky-type decomposition $S(a, b)$ for the corresponding interior solutions of (1) given by*

$$\begin{aligned} S(a, b) &\equiv \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} h \left(\frac{\partial}{\partial a} g\right) \\ &= \frac{\partial}{\partial a} h \Big|_{f=a} + \frac{\partial}{\partial b} h \left(\phi \left(\frac{\partial}{\partial a} g\right) - \mu \left(\frac{\partial}{\partial a} f\right)\right) \end{aligned}$$

(where ϕ and μ are defined in (18) and (19) below).[3] If $f \in D_u$ and $g \in E_u$ the result holds globally.

Proof. For a given f and g, a necessary condition for an interior maximum is given by (3) above. Locally, at the maximum, the differential associated to the map given by (3) can be written as

$$\begin{aligned} &\left(\frac{\partial}{\partial x} g\right) dx + \left(\frac{\partial}{\partial a} g\right) da - db = 0, \\ &\left(\frac{\partial^2}{\partial x^2} f\right) dx + \left(\frac{\partial^2}{\partial x \, \partial a} f\right) da + \left(\left(\frac{\partial^2}{\partial x^2} g\right) dx\right) \lambda \\ &\qquad + \left(\left(\frac{\partial^2}{\partial x \, \partial a} g\right) da\right) \lambda + \left(\frac{\partial}{\partial x} g\right) d\lambda = 0. \end{aligned} \tag{4}$$

[2] Generic existence of Z of (6) can be proved following the techniques of Theorem 1.

[3] Transposes of matrices are not indicated.

Let $L(x, a, \lambda, b)$ denote $f(x, a) + \lambda(g(x, a) - b)$. Then (4) can be written as

$$\begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix} \begin{pmatrix} d\lambda \\ dx \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial}{\partial a} g\right) da + db \\ -\left(\frac{\partial^2}{\partial x\,\partial a} L\right) da \end{pmatrix}. \tag{5}$$

Thus, for all f, a, b on an open and dense set of $D \times A \times B$, by the results of Theorem 1 and by (5),

$$\begin{pmatrix} d\lambda \\ dx \end{pmatrix} = \begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix}^{-1} \begin{pmatrix} -\left(\frac{\partial}{\partial a} g\right) da + db \\ -\left(\frac{\partial^2}{\partial x\,\partial a} L\right) da \end{pmatrix}. \tag{6}$$

By inverting a partitioned matrix we have

$$\begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix}^{-1}$$

$$= \left(\begin{array}{c|c} Z & -Z\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \\ \hline -\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z & \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \\ & \times \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \end{array}\right) \tag{7}$$

where

$$Z = -\left[\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right)\right]^{-1}. \tag{8}$$

From (6) and (7),

$$dx = \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\left(\frac{\partial}{\partial a} g\right) da - db\right)$$
$$- \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\,\partial a} L\right) da. \tag{9}$$

From (9),

$$\frac{\partial}{\partial a} x = \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial a} g\right)$$
$$- \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\,\partial a} L\right) \tag{10}$$

and

$$\frac{\partial}{\partial b} x = -\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z. \tag{11}$$

We now consider the effect of a "compensated" change in the vector a, obtained by a change in the parameter b, which keeps the value of the objective function constant, i.e., when

$$df = \left(\frac{\partial}{\partial x} f\right) dx + \left(\frac{\partial}{\partial a} f\right) da = 0.$$

From (3), this implies that at the maxima,

$$-\lambda \cdot \left(\frac{\partial}{\partial x} g\right) dx + \left(\frac{\partial}{\partial a} f\right) da = 0. \tag{12}$$

Also,

$$db = \left(\frac{\partial}{\partial x} g\right) dx + \left(\frac{\partial}{\partial a} g\right) da. \tag{13}$$

Hence, by (12) and (13) when $df = 0$,

$$-\lambda \left(db - \frac{\partial}{\partial a} g\, da\right) + \frac{\partial}{\partial a} f\, da = 0, \tag{14}$$

which implies that when $df = 0$, the db^i's are not all linearly independent. For instance, if b^i is the ith component of the vector b, in component form we get

$$\left(db^i - \left(\frac{\partial}{\partial a} g^i\right) da\right) = \frac{1}{\lambda_i} \left(\frac{\partial}{\partial a} f\right) da - \frac{1}{\lambda_i} \sum_{r \neq i} \lambda_r \left(db^r - \left(\frac{\partial}{\partial a} g^r\right) da\right). \tag{15}$$

Thus (14) and (15) imply that

$$db - ((\partial/\partial a)\, g)\, da, \qquad \text{when} \qquad df = 0, \tag{16}$$

becomes

$$\mu \left(\frac{\partial}{\partial a} f\right) da + \phi \left(db - \left(\frac{\partial}{\partial a} g\right) da\right), \tag{17}$$

where

$$\mu = (0, \ldots, 0, \overset{i\text{th place}}{1/\lambda_i}, \ldots, 0) \tag{18}$$

and

$$\begin{aligned} \phi &= (\phi_{q,r}), & & q = 1, \ldots, m, \quad r = 1, \ldots, m, \\ \phi_{q,r} &= 1 & &\text{if} \quad q = r, \quad q \neq i, \\ &= 0 & &\text{if} \quad q \neq r \quad \text{and} \quad q \neq i, \\ \phi_{i,r} &= -\frac{\lambda_r}{\lambda_i} & &\text{if} \quad r \neq i, \\ \phi_{i,i} &= 0, & &\text{and if} \quad b \in R, \quad \phi = 0. \end{aligned} \tag{19}$$

Therefore, from (9), (16), and (17) (denoting dx restricted to $f = c$, by $dx|_{f=c}$),

$$dx|_{f=c} = \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(-\mu\left(\frac{\partial}{\partial a}f\right)da - \phi\left(db - \left(\frac{\partial}{\partial a}g\right)da\right)\right)$$
$$- \left[\left(\frac{\partial^2}{\partial x^2}L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(\frac{\partial}{\partial x}g\right)\left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\,\partial a}L\right)da$$

and thus,

$$\frac{\partial}{\partial a}x\Big|_{f=c} = \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(-\mu\left(\frac{\partial}{\partial a}f\right) + \phi\left(\frac{\partial}{\partial a}g\right)\right)$$
$$- \left[\left(\frac{\partial^2}{\partial x^2}L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(\frac{\partial}{\partial x}g\right)\left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\,\partial a}L\right). \tag{20}$$

So, by (10), (11), and (20), we obtain

$$\frac{\partial}{\partial a}x + \frac{\partial}{\partial b}x\left(\frac{\partial}{\partial a}g\right)$$
$$= \frac{\partial}{\partial a}x\Big|_{f=c} + \frac{\partial}{\partial b}x\left(\phi\left(\frac{\partial}{\partial a}g\right) - \mu\left(\frac{\partial}{\partial a}f\right)\right)$$
$$= -\left[\left(\frac{\partial^2}{\partial x^2}L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(\frac{\partial}{\partial x}g\right)\left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\,\partial a}L\right)$$
$$\equiv S(a, b). \tag{21}$$

Since the x in (21) is by assumption a maximum, this completes the proof.

2

In Propositions 1 and 2 we study some conditions and forms of objective and constraint functions for which $S(a, b)$ has SN properties for the whole space of variables and parameters.

PROPOSITION 1. *Assume the objective function $f(x, a)$ has the form*

(i) $f \equiv \gamma[a \cdot x] + f^1(x) + f^2(a)$ *and the constraints $g(x, a)$ have the form*

(ii) $g^i \equiv \delta^i[a \cdot x] + g^{i1}(x) + g^{i2}(a)$, $i = 1, 2, \ldots, m$, *and the conditions of Theorem 2 of Section 1 are satisfied for f^1, f^2, and g^{i1}, g^{i2}. Then there exists a unique local C^1 solution for problemen (1) of Section 1, and the Slutsky-type decomposition is symmetric when $n = q$.*

Proof. In view of Theorem 2 we obtain

$$S(a, b) = -\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \left(\frac{\partial^2}{\partial x\, \partial a} L\right).$$

Computing

$$\left(\frac{\partial^2}{\partial x\, \partial a} L\right)$$

for the above objective and constraint functions we obtain

$$\left(\frac{\partial^2}{\partial x\, \partial a} L\right) = \begin{pmatrix} \gamma + \lambda\delta & & 0 \\ & \ddots & \\ 0 & & \gamma + \lambda\delta \end{pmatrix}.$$

This completes the proof.

PROPOSITION 2. *Under the conditions of Proposition 1 when $f \in D_0$ and $g \in E_0$, the Slutsky operator S is negative semidefinite if*

$$\gamma + \sum_i \lambda_i \delta^i \geqslant 0.$$

Proof. Negative semidefiniteness of the Slutsky operator S is obtained from the conditions for (i) and (ii) of Proposition 1 as follows. First we prove that

$$D = \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x} L\right)^{-1}\right]$$

is negative semidefinite.

Let z be any vector, and define a quadratic $Q_D = z'Dz$. Let $H = ((\partial^2/\partial x^2) L)$, and $H^{-1/2}$ be the symmetric negative square root of H^{-1}. Define $u = H^{-1/2}v$, where $v = (\partial/\partial x) g$, and $y = H^{-1/2}z$. Then,

$$\begin{aligned} Q_D &= y'y - y'u(u'u)^{-1} u'y \\ &= \| y \|^2 - \| u \|^{-2} \| u'y \|^2. \end{aligned}$$

By the Schwarz inequality, $Q_D \geqslant 0$. So, $S(a, b)$ will be negative semidefinite if $(\partial^2/\partial x\, \partial a) L$ is positive semidefinite since under the conditions of the proposition, $(\partial^2/\partial x\, \partial a) L$ is diagonal. But $(\partial^2/\partial x\, \partial a) L$ is positive semidefinite if $\gamma + \Sigma_i \lambda_i \delta^i \geqslant 0$. This completes the proof.

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