Journal of Mathematical Economics xxx (2009) xxx–xxx

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

The topology of fear☆

Graciela Chichilnisky*

Columbia University, 335 Riverside Drive, NY 10025, United States

ARTICLE INFO

Article history:
Received 24 February 2007
Received in revised form 4 June 2009
Accepted 4 June 2009
Available online xxx

Keywords:
Decision theory
Risk
Uncertainty
Catastrophes
Fear
Topology
Axioms of decision theory
Representation theory
Rare events
Equity premium puzzle
Allais paradox
Experimental work
Functional MRIs
Rare events
Catastrophes

ABSTRACT

For many years experimental observations have raised questions about the *rationality* of economic agents—for example, the Allais Paradox or the Equity Premium Puzzle. The problem is a narrow notion of rationality that disregards fear. This article extends the notion of rationality with new axioms of choice under uncertainty and the decision criteria they imply (Chichilnisky, G., 1996a. An axiomatic approach to sustainable development. Social Choice andWelfare 13, 257–321; Chichilnisky, G., 2000. An axiomatic approach to choice under uncertainty with Catastrophic risks. Resource and Energy Economics; Chichilnisky, G., 2002. Catastrophical Risk. Encyclopedia of Environmetrics, vol. 1. John Wiley & Sons, Ltd., Chicester). In the absence of catastrophes, the old and the new approach coincide, and both lead to standard expected utility. A sharp difference emerges when facing rare events with important consequences, or catastrophes. Theorem 1 establishes that a classic axiom of choice under uncertainty – Arrow's Monotone Continuity axiom, or its relatives introduced by DeGroot, Villegas, Hernstein and Milnor – postulate rational behavior that is 'insensitive' to rare events as defined in (Chichilnisky, G., 1996a. An axiomatic approach to sustainable development. Social Choice andWelfare 13, 257–321; Chichilnisky, G., 2000. An axiomatic approach to choice under uncertainty with Catastrophic risks. Resource and Energy Economics; Chichilnisky, G., 2002. Catastrophical Risk. Encyclopedia of Environmetrics, vol. 1. John Wiley & Sons, Ltd., Chicester). Theorem 2 replaces this axiom with another that allows extreme responses to extreme events, and characterizes the implied decision criteria as a combination of expected utility with extremal responses. Theorems 1 and 2 offer a new understanding of rationality consistent with previously unexplained observations about decisions involving rare and catastrophic events, decisions involving fear, the Equity Premium Puzzle, 'jump diffusion' processes and 'heavy tails', and it agrees with (Debreu, G., 1953. Valuation equilibrium and Pareto optimum. Proceedings of the National Academy of Sciences, 40, 588–592) formulation of market behavior and his proof of Adam Smith's Invisible Hand theorem.

1. Introduction

Catastrophes are rare events with major consequences. Examples are asteroids, earthquakes, market crashes or the 2008 global financial crisis originating from default in mortgages and extending more generally to global financial mar-

☆ This research is part of the Columbia's Program on Information and Resources, and the Columbia Consortium for Risk Management (CCRM). A first version of this paper was presented at the NBER Conference *Mathematical Economics: The Legacy of Gerard Debreu* at UC Berkeley, October 21, 2005.

* Corresponding author. Tel.: +1 212 678 1148.
E-mail addresses: gc9@columbia.edu, chichilnisky1@gmail.com.
URL: http://www.chichilnisky.com.

0304-4068/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jmateco.2009.06.006

kets (Chichilnisky, 1996a, 2006; Chichilnisky and Wu, 2006).[1] Although they are rare, catastrophes play a special role in our decision processes. Using magnetic resonance experimental psychologists have observed that our brains react differently to situations involving extreme fear. Certain regions of the brain – for example, the *amygdala*– often light up when a person makes decisions while confronted with events that inspire extreme fear (Le Doux, 1996).[2] Neurologists believe that such events alter cognitive processes and the behavior that could be otherwise expected. These observations seem relevant to the issue of rationality in decision making under uncertainty, and inspired the results presented in this article.

Allais (1988), Tversky and Wakker (1995), and Kahneman and Tversky (1979) have shown that the predictions of standard economic models based on rational behavior conflict with the experimental evidence on how we make decisions.[3] In this article we argue that the problem lies in the standard definition of rationality, which is narrow and exclusively based on testing whether or not we optimize *expected utility*. Expected utility has known limitations. It underestimates our responses to rare events no matter how catastrophic they may be (Chichilnisky, 1996a, 2000). This insensitivity has unintended consequences. It creates an illusion of 'irrational behavior' since what we anticipate does not agree with what we observe.

This article argues that the insensitivity of expected utility to rare events and the attendant inability to explain responses to events that invoke fear, are the source of many of the experimental paradoxes and failures of rationality that have been found over the years. Theorem 1 traces the problem to a classic axiom of choice under uncertainty that was developed half a century ago in somewhat different forms by von Neumann, Morgenstern, De Groot, Arrow, Hernstein, Milnor, and Villegas, and which is at the foundation of expected utility theory. Arrow calls this axiom Monotone Continuity (MC) (Arrow, 1971; Chichilnisky, 1998).[4] In all cases, the axiom requires that nearby stimuli should lead to nearby responses, which is reasonable enough. However the notion of 'nearby' that is used in these classic pieces is not innocent: Theorem 1 establishes that this notion implies a form of *insensitivity to rare events.* Indeed, when introducing the Monotone Continuity axiom, Arrow explains the requirements as follows: "If one action is derived from another by altering the consequences for states of the world on an event which is sufficiently small in this sense, the preference relation of that action with respect to any other given action should be unaltered."[5] Theorem 1 proves that in requiring this axiom, Arrow is requiring that altering arbitrarily the outcomes of two actions on common events of small probabilities, should not alter the ranking; this result holds when utilities are bounded, as in the case in Arrow (1971) and Chichilnisky (1996a, 2000, 2002)[6]. Theorem 1 proves that Monotone Continuity is a form of insensitivity to rare events as defined in Chichilnisky (1996a, 2000, 2002).[7] Therefore this theorem establishes that expected utility theory is insensitive to rare events due to the underlying axiom of Monotone Continuity[8]. The notion of 'nearby' implied by these axioms is explained by De Groot as follows:[9] "It distinguishes a probability distribution that is countably additive from one that is only finitely additive"[10] Indeed, it is now known that countably additive measures give rise to criteria of choice that are insensitive to rare events[11] (Chichilnisky, 2002, 2000, 1996a). This is the reason for the paradoxical behavior of expected utility theory, which has been found experimentally wanting time and time again since its introduction in the 1950s. Expected utility theory is based on axioms that guarantee insensitivity to rare events. The critical axiom is Monotone Continuity as defined in Arrow (1971) or its relatives.[12] But humans are sensitive to catastrophes, which are rare events. Therefore expected utility gives rise to a notion of rationality that is somewhat unnatural and contrary to the experimental evidence.

We need to define rational behavior more broadly, and more in tune with the way humans behave. One solution is to negate the suspect axiom of Monotone Continuity and its relatives, and require instead *sensitivity to rare events*. This is the approach followed here. Theorem 2 requires axioms of choice that include sensitivity to rare events. This is a representation theorem that characterizes decision criteria with the new set of axioms. Based on (Chichilnisky, 1996a, 2000, 2002) we prove that rational decisions maximize a combination of expected utility with other types of criteria that are sensitive to rare events. Theorem 2 shows that the former are countably additive distributions while the latter are finitely additive ones. Corollary 1 shows that in the absence of rare events, both approaches coincide. Therefore this work provides

---

[1] See also Chichilnisky and Gorbachev (2004, 2005), Chichilnisky (2000, 2002), Chichilnisky and Heal (1993) and Chichilnisky and Wu (2006).

[2] Not all rare events invoke fear, but situations involving extreme fear are generally infrequent.

[3] These works focus on the predictions of *expected utility* in general terms, and their findings led to the creation of *behavioral economics*.

[4] Hernstein and Milnor (1953) call his version of this Axiom 2, on p. 293, and De Groot calls it Assumption SP_4 (De Groot, 2004), on p. 73.

[5] Quoted from Arrow (1971, p. 48, paragraph 2) next to the definition of the "Monotone Continuity" axiom. The axiom of Monotone Continuity is the second axiom defined by Arrow for choice under uncertainty; the first axiom is defined on p. 47 paragraph 2 of (Arrow, 1971) and is "Ordering: The individual's choice among axioms can be represented by an ordering."

[6] Bounded utilities are a standard requirement—they are required by Arrow and many others because they are the only known way to avoid the so-called St. Petersburg Paradox that was discovered by Bernoulli, as discussed below in Footnote 11, see also (Arrow, 1971). Therefore in this article we restrict ourselves to bounded utilities to avoid the St. Petersburg's paradox. Arrow attributes the introduction of his Monotone Continuity axiom to Villegas (1964, p. 1789). Arrow makes this attribution to Villegas in footnote 2 to page 48 of (Arrow, 1971).

[7] By analyzing De Groot's Assumption SP_4 that is related to Arrow's Monotone Continuity, we show the connection with countably additive measures. De Grott writes directly "A property of this type distinguishes a probability distribution that is countably additive . . .". I have shown separately that countably additive distributions are insensitive to rare events when utilities are bounded, see Chichilnisky (2002, 2000).

[8] Or its equivalents, namely Axioms 2 in Hernstein and Milnor and SP_4 in De Groot, op. cit.

[9] Axiom $SPp_4$is defined in p. 73 of (De Groot, 2004).

[10] De Groot states, when discussing his axiom or assumption SP_4: "A property of this type distinguishes a probability distribution that is countably additive from one that is only finitely additive." quoted from p. 73 paragraph 4 of (De Groot, 2004).

[11] This statement was formally established in Chichilnisky (1996a, 2000, 2002).

[12] Its relatives are Axiom 2 in Hernstein and Milnor (1953) SP_4 in De Groot (2004), and a corresponding axiom in Villegas (1964).

Please cite this article in press as: Chichilnisky, G., The topology of fear. J. Math. Econ. (2009), doi:10.1016/j.jmateco.2009.06.006

an extension of classic decision theory. In the face of catastrophes, the new approach suggests new ways of evaluating risks.

As an illustration consider the rule of thumb 'maximize expected utility while minimizing the worst losses in the event of a catastrophe'. This rule is *inconsistent* with expected utility. Therefore any observer that anticipates expected utility optimization will be disappointed, and will believe that there is irrationality at stake. But this is not true. The rule is rational once we take into account rational responses to extreme events. It is consistent with what people do on an everyday basis, and with the experimental evidence on 'jump-diffusion processes' and 'heavy tails' distributions that are not well explained by standard theory.[13] Purely finitely additive measures have also been considered in the work of Brown and Lewis (1981) when studying myopic economic agents, and in Gilles' work on bubbles as equilibrium prices (Gilles, 1989).

The key to the new decision criteria that is identified here is to define 'nearby' in a way that is sensitive to rare events. This concept of 'nearby' is what I call a 'topology of fear,' one that can be sharply sensitive to catastrophes even if they are infrequent.[14] A similar topology was used in Debreu's 1953 formulation (Debreu, 1953) of Adam Smith's Invisible Hand theorem.[15] The rankings of lotteries that satisfy the new axioms for choice under uncertainty are a mix of 'countably additive measures' with 'finitely additive measures' with both parts present. This combination has not been used before except in Chichilnisky (1996a,b, 2000, 2002), Chichilnisky and Shmatov (2005) and Chichilnisky et al. (2005). These type of measures could play an important role in explaining how our brains respond to extreme risks.

2. Background and examples

Uncertainty is described by a system that is in one of several **states**, indexed by the real numbers with the standard Lebesgue measure μ, or alternatively by a bounded segment of real numbers with the uniform measure. In each state a **utility function** $u : R^n \to R$ ranks the **outcomes**, which are described by vectors in R^n.[16] When the probability associated with each state is given, a description of the utility achieved in each state is called a **lottery**.[17] In our context a **lottery** is a function $f : R \to R$ and the space of all lotteries is a function space L that we take to be the space of measurable and **essentially bounded functions**.[18] $L = L_\infty(R)$ with the standard norm $\|f\| = \operatorname{ess\,sup}_{x \in R} |f(x)|$.

Axioms for choice under uncertainty describe natural and self evident properties of choice. Continuity is a standard requirement that captures the notion that nearby stimuli give raise to nearby responses. However continuity depends on the notion of 'closeness' that is used. For example, in Arrow (1971, p. 48), two lotteries[19] are **close** to each other when they have different consequences in **small** events, which Arrow defines as follows "An event that is far out on a *vanishing sequence* is **'small'** by any reasonable standards" (Arrow 1971, p. 48).[20] Our definition of closeness is quite different, and uses a L_∞ sup norm that is based on uniform proximity.[21]

A ranking function $W : L_\infty \to R$ is called **insensitive to "rare" events** when it neglects small "probability" events; formally if given two lotteries (f, g) there exists $\epsilon = \epsilon(f, g) > 0$, such that $W(f) > W(g)$ if and only if $W(f') > W(g')$ for all f', g' satisfying $f' = f$ and $g' = g$ a.e. on $A \subset R$ when $\mu(A^c) < \epsilon$.[22] Similarly, $W : L \to R$ is said to be **insensitive to "frequent" events** when for every two lotteries f, g there exists $\delta(f, g) > 0$ that $W(f) > W(g)$ if and only if $W(f') > W(g')$ for all f', g' such that $f' = f$ and $g' = g$ a.e. on $A \subset R : \mu(A^c) > \delta$.

We say that W is **sensitive to "rare" events**, when W is **not** insensitive to rare events as defined above, and we say that W is **sensitive to "frequent" events** when W is **not** insensitive to frequent events as defined above. The ranking W is called **continuous and linear** when it defines a linear function on the utility of lotteries that is continuous with respect to the norm in L_∞. Here are the new axioms introduced in Chichilnisky (2000, 2002):

Axiom 1. The ranking $W : L_\infty \to R$ is linear and continuous on lotteries

[13] In practical terms the criteria we propose can help explain experimental observations that conflict with the standard notions of rationality: the Allais paradox ((Chichilnisky, 1996a, 2000, 2002)), the Equity Premium and the Risk Free Rate Puzzles (Mehra and Prescott, 1985; Mehra, 2003; Weil, 1989), (Chichilnisky et al., 2005), and the prevalence of *heavy tails* and jump-diffusion processes in financial markets (Chichilnisky and Shmatov, 2005).

[14] This refers to the sup norm in L_∞.

[15] Debreu's article was submitted to the National Academy of Sciences by John Von Neumann, but its formulation goes much beyond Von Neumann's own axioms and his own formulation of expected utility.

[16] When more is better, the function $u : R^n \to R$ is a monotonically increasing continuous function.

[17] The definition of a *lottery* can take several forms, all leading to similar results. It can be defined by probability distributions given the outcomes in each state, or by outcomes rather than by the utility of these outcomes as we do here. We adopt this definition to simplify the presentation.

[18] *Boundedness* of the utility functions is critical. It is required both for theoretical and empirical reasons, in particular to fit observed behavior, as explained by Arrow (1971) to avoid the St. Peterburg's Paradox, discovered by Bernoulli, where people in St. Peterburg only wished to pay a finite amount of money to play a lottery that had an infinite expected value, see also De Groot (2004), Chapter 7, "Bounded Utility Functions.". The space of lotteries used here is the same $L_\infty(R)$ space used in 1953 by Debreu and others to identify commodities in infinitely many states (Debreu, 1953; Chichilnisky and Kalman, 1980). **Choice under uncertainty** means the ranking of lotteries. An **event** E is a set of states. E^c is the set of states of the world not in E. A **monotone decreasing** sequence of events $\{E^i\}_{i=1}^{\infty}$, is a sequence for which for all i, $E^{i+1} \subset E^i$. If there is no state on the world common to all members of the sequence, $\bigcap_{i=1}^{\infty} E^i = \varnothing$, $\{E^i\}$ is called a **vanishing sequence**

[19] The equivalent to the notion of "lotteries" in our framework is the notion of "actions" as defined in Arrow (1971).

[20] The formal definition of Monotone Continuity in Arrow (1971) is given in Appendix A.

[21] In Arrow's definition of closeness, two lotteries are close when they differ in sets of small probability. Infrequent events matter little under his definition.

[22] A^c denotes the complement of the set A.

Axiom 2. The ranking $W : L_\infty \to R$ is sensitive to rare events

Axiom 3. The ranking $W : L_\infty \to R$ is sensitive to frequent events

The **expected utility** of a lottery f is a ranking defined by $W(f) = \int_{x \in R} f(x) d\mu(x)$ where μ is a measure with an integrable density function $\phi(.)\epsilon L_1$ [23] so $\mu(A) = \int_A \phi(x)dx$, where dx is the standard Lebesgue measure on R. Expected utility satisfies Axioms 1 and 3, but not Axiom 2:

Example 1. *Expected utility is insensitive to rare events.*

For a proof see (Chichilnisky, 1996a, 2000, 2002).

3. Two approaches to decision theory

This section compares our approach with the classic theory of choice under uncertainty. Both approaches postulate that nearby stimuli lead to nearby responses or choices. There are however different views of what constitutes 'nearby.' The classic theory considers two lotteries to be close when they differ in events of small measure, while our notion is stricter, requiring that the lotteries be close almost everywhere, which implies sensitivity to rare events. See Example 6 in the Appendix.

The best way to compare the two approaches is to put side by side the decision criteria that each implies. The classic theory of choice under uncertainty, as presented in Arrow (1971) and Hernstein and Milnor (1953) shows that, on the basis of standard axioms of choice, the ranking of lotteries $W(f)$ is an *expected utility function*.[24]

Our decision criteria are different. Expected utility rankings do not satisfy our sensitivity Axiom 2, as shown in Example 1.[25] We need to modify expected utility adding another component called 'purely finitely additive' elements of L_∞^*.[26] The latter embody the notion of sensitivity for rare events, and identifies the 'topology of fear'. Purely finitely additive measures are the main difference between our decision criteria and the classic expected utility criterion.

Where exactly does this difference originate? It is possible to trace the difference between the two approaches to a single classic axioms of choice, the Axiom of Monotone Continuity (MC) stated formally in the Appendix, showing that this contradicts our sensitivity Axiom 2:

Theorem 1. *A ranking of lotteries $W(f) : L_\infty \to R$ satisfies the Monotone Continuity (MC) Axiom if and only if it is insensitive to rare events. Formally MC $\Leftrightarrow \sim$ Axiom 2 above.*

Proof. First we show in a simple case why the Monotone Continuity Axiom (MC) implies insensitivity to rare events, namely the negation of Axiom 2. Assume that MC is satisfied. By definition, MC implies that for every two lotteries $f \succ g$, every outcome c and every vanishing sequence of events $\{E^i\}$ there exists N such that altering arbitrarily the outcomes of the lotteries f and g on the event E^i, where $i > N$, does not alter the ranking—namely $f' \succ g'$, where f' and g' are the altered versions of lotteries f and g respectively.[27] In particular since, for any given f and g, Axiom MC applies to every vanishing sequence of events $\{E^i\}$, we may choose a sequence of events consisting of open intervals $I = \{I^i\}_{i=1}^\infty$ such that $I^i = \{x \in R : x > i\}$ and another $J == \{J^i\}_{i=1}^\infty$ such that $J^i = \{x \in R : x < -i\}$. Consider the sequence $K = \{K^i\}$ where $K^i = I^i \bigcup J^i$. The sequence K is a vanishing sequence by construction. Therefore there exists an $i > 0$ such that for all $N > i$, any alteration of the lotteries f and g over the set K^N, denoted f^N and g^N respectively, leave the ranking unchanged i.e. $f^N \succ g^N$. Therefore the MC Axiom implies insensitivity of the ranking W in unbounded sets of events such as $\{E^i\}$.

More generally, we will show that if the ranking satisfies Axiom 2 above, namely it is sensitive to rare events, W cannot satisfy MC. Axiom 2 implies that there at least are two lotteries f, g with $f \succ g$, and for these two lotteries a sequence of measurable sets $\{U^n\}_{n=1}^\infty$ such that for all $n = 1, 2 \ldots, (i) \mu(U^n) < 1/n$, where μ is the Lebesgue measure on R, and (ii) altering the lotteries f and g on the set U^n reverses the ranking namely $f^n \prec g^n$, where f^n and g^n are the lotteries resulting from the alteration of f and g on the set U^n. Now, without loss of generality we may assume for every $n = 1, \ldots,$ the sets in the sequence $\{U^n\}_{n=1}^\infty$ satisfy $U^{n+1} \subset U^n$. Otherwise, construct another sequence $\{V^n\}$, where $V^n = \bigcup_{j=1}^\infty U^j - \bigcup_{j=i}^n U^j$, such that the sequence $\{V^n\}$ has the ranking reversal property (ii); by choosing appropriately a subsequence of $\{V^n\}$, denoted

[23] $L_1(R)$ is the space of all measurable and integrable functions on $R, f \epsilon L_1 \Leftrightarrow \|f\| = \int_R |f(x)|dx < \infty$

[24] Expected utility functions are linear functions defined on lotteries and can be identified with countably additive measures. Here the expected utility criterion means that there exists a **bounded** utility function $u : R \to R$ and a probability density $\delta(t) : R \to R$, $\int_R \delta(t)dx = 1$ such that f is preferred over g if and only if $W(f) > W(g)$, where $W(f) = \int_R u(f(t), t)d\delta(t)$. In our framework, the classic axioms imply that the ranking $W(.)$ is a continuous linear function on L_∞ that is defined by a countably additive measure with an integrable density function $\delta(t)$ i.e. δ is in $L_1(R)$.

[25] The space L_∞^* is called the 'dual space' of L_∞, and is known to contain two different types of rankings $W(.)$, (i) integrable functions in $L_1(R)$ that can be represented by countably additive measures on R, and (ii) 'purely finitely additive measures'which are not representable by functions in L_1 (Chichilnisky, 2000), cf. the Appendix.

[26] Indeed, there is an entire subspace of L_∞^* that consists of functions that do not have a representation as L_1 functions in R. This subspace consists of 'purely finitely additive measures' that are defined in the Appendix, with examples provided there. Purely finitely additive measures are not representable by countably additive measures, they are not continuous with respect to the Lebesgue measure of R, and they cannot be represented as functions in $L_1(R)$.

[27] For simplicity, one may consider alterations in those lotteries that involve the 'worst' outcome $c = \inf_R |f(x), g(x)|$, which exists because f and g are bounded a.e. on R by assumption.

Please cite this article in press as: Chichilnisky, G., The topology of fear. J. Math. Econ. (2009), doi:10.1016/j.jmateco.2009.06.006

also $\{V^n\}$, we can ensure that $\mu(V^n) < 1/n$. Observe that this new sequence $\{V^n\}$ also satisfies (i), (ii), and by construction for all n, $V^{n+1} \subset V^n$. Furthermore, without loss of generality we may also assume (iii) the family of sets $\{V^n\}$ has empty intersection, i.e. $\bigcap_{n=1}^{\infty} V^n = \varnothing$. This is because the intersection of the family $\{V^n\}$, $\bigcap_{n=1}^{\infty} V^n$, has Lebesgue measure zero since, by construction, for all n, $\mu(V^n) < 1/n$. Since by construction the lotteries f and g are both elements of L_∞, and are therefore defined almost everywhere in the Lebesgue measure in R, we may therefore consider a new family $\{W^n\}$ such that $W^n = V^n - (\bigcap_{n=1}^{\infty} V^n)$ without changing the lotteries f and g in any way, in particular keeping the properties (i) and (ii). Because the family V satisfies (ii) and (iii) it satisfies the definition of a vanishing family. If the Axiom of Monotone Continuity (MC) was satisfied, then for some n, and all $N > n$, altering the lotteries f and g on the set V^N leads to $f^N \succ g^N$, which is a contradiction with (ii). Therefore we have shown that whenever Axiom 2 is satisfied, the Axiom of Monotone Continuity (MC) cannot hold. This completes the first part of the proof, namely that $MC \Rightarrow \sim$ Axiom 2.

Conversely, we will establish that the negation of Axiom MC implies Axiom 2. Assume now that Axiom MC is not satisfied. Then for some pair of lotteries f, g and outcome c there exists a vanishing sequence $\{E^i\}$ and for every $N > 0$, there is an $i > N$ such that $f \succ g$, while arbitrary alterations of f and g on E^i denoted f' and g' reverse the ranking, namely $g^i \succ f^i$. Observe that since $\{E^i\}$ is a vanishing sequence and μ the Lebesgue measure on R, $lim_{N\to\infty}\mu(E^N) = 0$. Since we assumed that Axiom MC is not satisfied, this implies that altering f and g on sets of arbitrarily small measure reverses the ranking, i.e. $g^i \succ f^i$. Since this holds for all $N > 0$, there is no $\varepsilon = \varepsilon(f, g) > 0$ such that $f \succ g \Leftrightarrow f^i \succ g^i$ for any f' and g' that are arbitrary modifications of f and g on sets of measure smaller than ε. Thus, by definition, the ranking W is sensitive to rare events or equivalently, Axiom 2 is satisfied. We have therefore shown that the negation of Axiom MC implies $\sim$ Axiom 2, or equivalently that the negation of Axiom MC implies Axiom 2. The Monotone Continuity Axiom MC is therefore equivalent to insensitivity to rare events, as we wished to prove. □

The following representation theorem identifies all the decision criteria that satisfy Axioms 1, 2 and negates MC. Examples are provided in the Appendix.

Theorem 2 (*A ranking of lotteries*). *$W : L_\infty \to R$ satisfies Axioms 1, 3 and $\sim$MC if and only if there exist two continuous linear functions on L_∞, ϕ_1 and ϕ_2 and a real number λ, $0 < \lambda < 1$, such that:*

$$W(f) = \lambda \int_{x\in R} f(x)\phi_1(x)dx + (1-\lambda)\langle f, \phi_2\rangle \tag{1}$$

where $\int_R \phi_1(x)dx = 1$, while ϕ_2 is a purely finitely additive measure. [28]

Proof. This follows from Theorem 1 since the set of *Axioms 1, 3 and* $\sim$ MC is equivalent to the set of Axioms1, 2 and 3 since by Theorem 1 Axiom 2 is equivalent to the negation of MC. Now apply the representation theorem in Chichilnisky (1996b, 2000, 2002) to establish that a ranking that satisfies Axioms 1, 2 and 3 must have the form 1. □

Observe that the first term in (1) is similar to an expected utility, where the density function could be for example $\phi_1(x) = e^{-\delta x}$. Such a density defines a countably additive measure that is absolutely continuous with respect to the Lebesgue measure.[29] The second term of (1) is of a different type: the operator $\langle f, \phi_2\rangle$ represents the action of a measure $\phi_2 \epsilon L^*_\infty$ that differs from the Lebesgue measure as it puts all weight on rare events.

The following *rules of thumb* provide an intuitive illustration of the decision criteria that satisfy all three axioms invoked in Theorem 2. The actual examples are provided in the Appendix:

Example 2. Choosing a portfolio: Maximize expected utility while seeking to minimize total value losses in the event of a catastrophe.

Example 3. Network Optimization: Maximize expected through-put of electricity while seeking to minimize the probability of a black-out.

Example 4. Heavy Tails: The following function illustrates the singular measure that appears in the second term in (1) a special case, for those lotteries in L_∞ that have limiting values at infinity, $L'_\infty = \{f \epsilon L_\infty : \lim_{x\to\infty}(x) < \infty\}$. Define

$$\Psi(f) = \lim_{x\to\infty} f(x) \tag{2}$$

Ψ is a continuous linear function on L_∞ that is not representable by an L_∞ function as in (1); it is also insensitive to events that are frequent with respect to the Lebesgue measure because it only takes into consideration limiting behavior, and does not satisfy Axiom 3. This asymptotic behavior tallies with the observation of *heavy tails* in financial markets (Chichilnisky and Shmatov, 2005). Observe that the function Ψ is only defined on a subspace of L_∞; to define a purely finitely additive measure on all of L_∞ one seeks to extend Ψ to all of L_∞. The last section of this article describes what is involved in obtaining such an extension.

[28] Definitions of 'countably additive' and 'purely finitely additive' measures are in the Appendix under the heading *The dual space L^*_∞*. Observe that ϕ_2 as defined above cannot be represented by an L_1 function.

[29] A measure is called *absolutely continuous* with respect to the Lebesgue measure when it asigns zero measure to any set of Lebesgue measure zero; otherwise the measure is called *singular*.

If one considers a family of subsets of events containing no rare events, for example when the Lebesgue measure of all events contemplated is bounded below, then the two approaches are identical:

Corollary 1. *Absent rare events, a ranking that satisfies Axioms 1, 2, and 3 is consistent with Expected Utility Theory.*

Proof. Observe that Axiom 2 is an empty requirement when there are no rare events, and Axioms 1 and 3 are consistent with Expected Utility. □

4. The topology of fear and the value of life

Our axioms and Debreu's (1953) work use the same notion of continuity or 'closeness' of lotteries: **Proximity** of two lotteries requires that the **supremum** of their distance across all states should be small almost everywhere (a.e.). Distance is therefore measured by extremals. The formal description of this topology is the standard sup norm of L_∞. Since the topology focuses on situations involving extremal events, such as catastrophes, it makes sense to call this the **topology of fear**.

Our Axiom 1 requires continuity of the ranking,[30] and it **is** satisfied by expected utility functions. However our Axiom 2 requires a further condition, that W be sensitive to rare events. Expected utility does not satisfy this condition as shown in Theorem 1 above, because expected utility satisfies Axiom MC that is equivalent to the negation of Axiom 2.[31]

One requirement of the classic theory, Axiom MC defined in the Appendix, is key because it creates insensitivity to rare events (Theorem 1). We can illustrate how this works in the following situation that was pointed out by Arrow (Arrow, 1971) about how people value their lives.

4.1. How people value their lives

The following example is from Arrow (1971, pp. 48–49), see also Chichilnisky (1998, pp. 257–258). If a is an action that involves receiving one cent, b is another that involves receiving zero cents, and c is a third action involving receiving one cent and facing a small probability of death, Arrow's *Monotone Continuity* requires that the third action involving death and one cent should be preferred to the second involving zero cents when the probability of death is small enough. Even Kenneth Arrow says of his requirement 'this may sound outreageous at first blush ...' (Arrow, 1971). Outrageous or not, we saw in Theorem 1 that MC leads to the neglect of rare events with major consequences, like death.

Theorem 1 shows that our Axiom 2 rules out these examples that Arrow calls 'outrageous', because Axion 2 is the negation of MC. We can also exemplify how our Axiom 2 provides a reasonable resolution to the problem, as follows. Axiom 2 implies that there exist catastrophic outcomes such as the risk of death, so terrible that one is unwilling to face a small probability of death to obtain one cent versus half a cent, no matter how small the probability may be. Indeed, according to our sensitivity Axiom 2, no probability of death is acceptable when one cent and half a cent are involved. However according to Axiom 2, in other cases, there may be a small enough probability that the lottery involving death may be acceptable. It all depends on what are the other outcomes involved. For example, if instead of one cent and zero cents one considers one billion dollars and zero cents – as Arrow suggests in Arrow (1971) – under certain conditions one may be willing to take the lottery that involves a small probability of death and one billion dollars over the one that offers half a cent. More to the point, a small probability of death caused by a medicine that can cure an incurable cancer may be preferable to no cure. This seems a reasonable solution to the issue that Arrow raises. Sometimes one is willing to take a risk with a small enough probability of a catastrophe, in other cases one is not. It all depends on what else is involved. This is the content of our Axiom 2.

In any case, absent rare events with major consequences, our Axiom 2 is voided of meaning. In this case our theory of choice under uncertainty is consistent with or collapses to the standard expected utility theory. Therefore this work can be viewed as an extension of classic decision theory.

5. Gerard Debreu and the invisible hand

In establishing a mathematical proof of Adam Smith's Invisible Hand theorem in 1953, Debreu (1953) was not explicitly concerned with rare or extremal events. The goal of his article was to show that any Pareto efficient allocation can be decentralized as the equilibrium of a competitive market. This is the major conclusion from Adam Smith's Invisible Hand Theorem. Yet in his article, (Debreu, 1953) for markets with infinitely many commodities, such as dynamical models used in finance, Debreu proposed that a natural space of commodities would be the space of essentially bounded measurable functions L_∞ that we use in this article to describe lotteries. Debreu used the same notion or proximity, or 'nearby' lotteries, the *supnorm* that we use here. Debreu chose this space because it is alone among all infinite dimensional L_p spaces in having a positive quadrant with a non - empty interior. There is a profound mathematical wisdom behind this choice. The property

[30] With respect to the sup norm.

[31] As already pointed out, Arrow (1971, p. 257), introduced the axiom of *Monotone Continuity* attributing it originally to Villegas (1964, p. 1789). It requires that modifying an action in events of small probabilities should lead to similar rankings. At the same time (Hernstein and Milnor, 1953, p. 293) require a form of continuity in their Axiom 2 that is similar to Arrow's *Monotone Continuity* and leads to their *Continuity Theorem* on p. 293. The axioms of continuity required by Arrow and by Hernstein and Milnor are quite different from the type of continuity that we require here, see the Appendix Example 7.

of positive orthants just quoted is crucial to his proof of decentralization by market prices, which relies on Hahn–Banach's separating hyperplane theorem, a theorem that requires the existence of interior or internal points for otherwise it is not valid, see Chichilnisky (1977), and Chichilnisky and Kalman (1980).

Debreu's choice of commodity space in Debreu (1953) leads naturally to Pareto efficient allocations where prices are in the dual of L_∞ namely in L_∞^* which, as we saw above, contain both standard L_1 functions *as well as* purely finitely additive measures that are not representable by standard functions, see examples in the Appendix. In the latter case, one may lose the ability of assigning a price to a commodity within a given period of time, or within a given state of uncertainty. A non - zero price can give rise to zero value in each period and each state. Examples of this nature were constructed by Malinvaud (1953) and later on by McFadden (1975) and were extensively discussed in the literature (Radner, 1965; Majumdar and Radner, 1972; Chichilnisky and Kalman, 1980). In 1980 this author and Peter Kalman provided a necessary and sufficient condition to overcome the problem, a 'cone condition' that is necessary and sufficient for the prices that support Pareto Efficient allocations to be always in L_1, Chichilnisky and Kalman (1980) and Chichilnisky (1993). A few years later, this result was followed by a similar *uniform properness* condition used in Mas Collel (1986). The results in Chichilnisky and Kalman (1980); Chichilnisky (1993) and later on by Mas Collel (1986) modified the original L_∞ model introduced by Debreu and ensured that all Pareto efficient allocations would have associated market prices that are well defined in every state and every time period. They resolved the paradoxical behavior of the infinite dimensional model that Debreu had introduced in 1953, by eliminatinf finitely additive measures.

Debreu's original 1953 formulation was however prescient.

Sixty years later, we find that the general L_∞ framework that he proposed in 1953 is consistent with the decision criteria emerging from the new axioms for choice under uncertainty presented here, giving rise to prices in L_∞^* that are partly in L_1 and partly not, as shown in the representation Theorem 2 above. The L_1 part overcomes the concerns expressed above, since it creates non-zero value in each period and in each state. The purely finitely part allows sensitivity to *rare events.* This solution was certainly not contemplated by the literature that followed Debreu's 1953 work, which focused instead on eliminating the purely finitely additive parts. This elimination is not necessary. One can take into account – as shown above – that the criteria may have two different parts, and that one of them (in L_1) suffices to define non-zero prices in all states and periods.

In any case, Gerard Debreu's 1953 theorem on the Invisible Hand is compatible with the decision criteria postulated in this paper. And while such decision criteria could have seemed paradoxical at the time, it seems now that they may be better suited to explain the experimental evidence and the observations of behavior under uncertainty than expected utility, when catastrophic events are at stake.

6. The Axiom of choice and rare events

There is another interesting mathematical connection to the axioms for choice under uncertainty presented in this article. This connection is, naturally enough, with the venerable Axiom of Choice in the foundation of Mathematics. The Axiom of Choice postulates that there exists a universal and consistent fashion to select an element from every set. This section illustrates how it is connected with the axioms of choice under uncertainty that are proposed here.

The best way to describe the situation is by means of an example, see also (Yosida and Hewitt, 1952; Yosida, 1974; Chichilnisky and Kalman, 1980).

6.1. Representing a purely finitely additive measure

Consider the purely finitely measure ρ defined in the Appendix Example 7: for every Borel measurable set $A \subset R$, $\rho(A) = 1$ whenever $A \supseteq \{r : r > a$, for some $a \in R\}$, and $\rho(A) = 0$ otherwise. It is easy to show that ρ is not countably additive.

Consider a family of countably many disjoint sets $\{V_i\}_{i=0,1,\ldots}$ defined as follows

$$\text{for } i = 0, 1, \ldots, \qquad V_i = (i, i+1) \bigcup (-i-1, -i]$$

Observe that any two sets in this family are disjoint namely

$$V_i \bigcap V_i = \varnothing \text{ when } i \neq j,$$

and that the union of the entire family covers the real line:

$$\bigcup_{i=0}^{\infty} V_i = \bigcup_{i=0}^{\infty} (i, i+1) \bigcup (-i-1, -i] = R.$$

Since the family $\{V_i\}_{i=0,1,\ldots}$ covers the real line, the measure of its union is one, i.e.

$$\rho(\bigcup_{i=0}^{\infty} V_i) = 1. \tag{3}$$

Yet since every set V_i is bounded, each set has ρ, measure zero by definition. If ρ was countably additive we would have

$$\rho(\bigcup_{i=0}^{\infty} V_i) = \sum_{i=0}^{\infty} V_i = 0,$$

which contradicts (3) Since the contradiction arises from assuming that ρ is countably additive, ρ must be purely finitely additive, as we wished to prove.

Observe that ρ assigns zero measure to bounded sets, and a positive measure only to unbounded that contain a '*neighborhood of* $\{\infty\}$.[32] We can define an explicit function on L_∞ that represents the action of this purely finitely additive measure ρ if we restrict our attention to the closed subspace L'_∞ of L_∞ consisting of those functions $f(x)$ in L_∞ that have a limit when $x \to \infty$, by the formula

$$\rho(f) = \lim_{x\to\infty} f(x) \tag{4}$$

When restricted to functions f in the subspace L'_∞

$$\rho(f) = \int f(x) d\rho(x) = \lim_{x\to\infty} f(x).$$

Observe that one can describe the function ρ as a limit of a sequence of delta functions whose support increases without bound:

$$\rho(f) = \lim_{N\to\infty} \int f(x).\Delta_N = \lim_{N\to\infty} f(N)$$

where $\{N\}_{N=1,2,\ldots}$ is the sequence of natural numbers, and where

$$\Delta_N \text{ is a ' delta' measure on } R \text{ supported on the set} \{N\},$$

$$\text{defined by} \Delta_N(A) = 1 \text{ when } A \supset (N-\varepsilon, N+\varepsilon) \text{ for some } \varepsilon > 0,$$

$$\text{and} \Delta_N(A) = 0 \text{otherwise}.$$

The problem is now to extend the function ρ to a function that is defined on the entire space L_∞. This could be achieved in various ways but as we will see, each of them requires the Axiom of Choice.

One way is to use Hahn–Banach's theorem to extend the function ρ from the closed subspace $L'_\infty \subset L_\infty$ to the entire space L_∞ while preserving its norm.[33] However, in its general form Hahn–Banach's theorem requires the Axiom of Choice. Alternatively, one can extend the notion of a *limit* in (4) to encompass all functions in L_∞ including those that have no limit. This can be achieved by using the notion of convergence along a *free ultrafilter* arising from compactifying the real line R as in (Chichilnisky and Heal, 1997). However the existence of a *free ultrafilter* requires the Axiom of Choice.

This illustrates why the attempts to construct *purely finitely additive measures* that are representable as functions on L_∞, require the Axiom of Choice. Since our criteria require purely finitely additive measures, this illustrates a connection between the Axiom of Choice and our axioms for choice under uncertainty. This connection is not entirely surprising since both sets of axioms are about the ability to choose, universally and consistenly. What is interesting, however, is that the consideration of rare events that are neglected in standard decision theory conjures up the Axiom of Choice.

Acknowledgements

I am grateful to Kenneth Arrow, Bernard Cornet, Jean Michel Grandmont, Peter Hammond, Werner Hildenbrand, Andreu Mas Colell, Marcel Richter and other participants of the 2007 NBER General Equilibrium Conference at UC Berkeley in honor of Gerard Debreu for valuable discussions. An anonymous referee provided insightful comments that improved the paper.

[32] By a 'neighborhood of ∞' we mean the sets of the form

$$\{x \epsilon R : x > a \text{for some} a \epsilon R\}.$$

[33] The norm of the function $\lim f(x)$ defined above is one, because by definition

$$\| \lim(.)\| \sup_{f \epsilon L, \|f\|=1} |\lim f(x)| = 1.$$

.

Please cite this article in press as: Chichilnisky, G., The topology of fear. J. Math. Econ. (2009), doi:10.1016/j.jmateco.2009.06.006

Appendix A.

A.1. Arrow's definition of Monotone Continuity (MC)

The following is the definition of continuity used in Arrow (1971, p. 48) A vanishing sequence of sets is one whose intersection is empty. For example, in the case of the real line, the sequence $\{(n, \infty)\}$, $n = 1, 2, 3 \ldots$, is a vanishing sequence of sets. The following is Arrow's **Axiom of Monotone Continuity** (**MC**): Given a and b, where $a \succ b$, a consequence c, and a vanishing sequence $\{E^i\}$, suppose the sequences of actions $\{a^i\}$, $\{b^i\}$ satisfy the conditions that (a^i, s) yield the same consequences as (a, s) for all s in $(E^i)^c$ and the consequence c for all s in E^i, while (b^i, s) yields the same consequences as (b, s) for all s in $(E^i)^c$ and the consequence c for all s in E^i. Then for all i sufficiently large, $a^i \succ b$ and $a \succ b^i$."

Example 5. A ranking that is insensitive to frequent events.

Consider $W(f) = \lim\inf_{x \in R}(f(x))$. This ranking is insensitive to frequent events of arbitrarily large Lebesgue measure (see (Chichilnisky, 2000)) and therefore does not satisfy our Axiom 3. In addition, this ranking is not linear.

Example 6. Two approaches to the 'closeness' of lotteries.

This example provides two sequence of lotteries that converge to each other according to the notion of closeness defined in Arrow (1971), but not according to our notion of closeness using the sup norm. Our notion of closeness is thus more demanding, implying our requirement of sensitivity to rare events.

Arrow's Axiom MC requires that if a lottery is derived from another by *altering its outcomes* on events that are *sufficiently small* (as defined in (Arrow, 1971, p. 48)), the preference relation of that lottery with respect to any other given lottery should remain unaltered.[34] To define 'small' differences between two lotteries, or equivalently to define 'closeness', Arrow considers two lotteries (f, g) to be 'close' when there is a vanishing sequence $\{E^i\}$, such that f and g differ only in sets of events E^i for large enough i (Arrow, 1971, p. 48). This notion of closeness is based on the standard Lebesgue measure. Observe that for any vanishing sequence of events $\{E^i\}$, as i becomes large enough, the Lebesgue measure μ of the set E^i becomes small, formally $\lim_{i \to \infty} \mu(E^i) = 0$. Therefore in Arrow's framework, two lotteries that differ in sets of small enough Lebesgue measure are very close to each other. Our framework is different since two lotteries f and g are 'close' when they are uniformly close almost everywhere, i.e. when $\sup_R |f(t) - g(t)| < \varepsilon a.e.$ for a suitably small $\varepsilon > 0$. The difference between the two concepts of 'closenes' is sharpest when considering vanishing sequences of sets. In our case, two lotteries that differ in sets of events along a vanishing sequence E^i may never get close to each other. Consider as an example the family $\{E^i\}$ where $E^i = [i, \infty)$, $i = 1, 2, \ldots$. This is a vanishing family of events, because $E^i \subset E^{i+1}$ and $\bigcap_{i=1}^{\infty} E^i = \varnothing$. Consider now the lotteries $f^i(t) = K$ when $t \in E^i$ and $f^i(t) = 0$ otherwise , and $g^i(t) = 2K$ when $t \in E^i$ and $g^i(t) = 0$ otherwise. Then for all i, $\sup_{E^i} |f^i(t) - g^i(t)| = K$. In our topology this implies that f^i and g^i are **not** 'close' to each other, as the difference $f^i - g^i$ does not converge to zero. No matter how far along we are along the vanishing sequence E^i the two lotteries f^i, g^i differ by K. Yet since the lotteries f^i, g^i differ from $f \equiv 0$ and $g \equiv 0$ respectively only in the set E^i, and $\{E^i\}$ is a vanishing sequence, for large enough i they are as 'close' as desired according to Arrow's (1971) definition. But they are not 'close' according to our notion of closeness, which is more demanding, as we wished to show.

A.2. The dual space L_∞^*

A.2.1. Countably and purely finitely additive measures

The space of continuous linear functions on L_∞ is a well known space called the "dual' of L_∞, and is denoted L_∞^*. This dual space has been fully characterized, e.g. in (Yosida and Hewitt, 1952; Yosida, 1974). Its elements are defined by integration with respect to measures on R. The dual space L_∞^* consists of $(i) L_1$ functions g that define countably additive measures μ on R by the rule

$$\mu(A) = \int_A g(x)dx$$

where $\int_R |g(x)|dx < \infty$ and therefore μ is *absolutely continuous* with respect to the Lebesgue measure, namely it gives measure zero to any set with Lebesgue measure zero, and (ii) a 'non- L_1 part' consisting of purely finitely additive measures ρ that are 'singular' with respect to the Lebesgue measure and give positive measure to sets of Lebesgue measure zero; these measures ρ are finitely additive but they are *not* countably additive. A measure η is called *finitely additive* when for any family of pairwise disjoint measurable sets $\{A_i\}_{i=1,\ldots N}$ $\eta(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \eta(A_i)$. The measure η is called *countably additive* when

[34] This statement is a direct quote Arrow's own explanation of the Axiom of Monotone Continuity, see (Arrow, 1971, p. 48), para 2., once one translates the notion of 'choosing an action' to the notion of 'choosing a lottery'.

for any family of pairwise disjoint measurable sets $\{A_i\}_{i=1,\ldots\infty}\eta(\bigcup_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}\eta(A_i)$. The countably additive measures are in a one to one correspondence with the elements of the space $L_1(R)$ of integrable functions on R (Yosida and Hewitt, 1952; Yosida, 1974). However, purely finitely additive measures cannot be identified by such functions. Nevertheless, purely finitely additive measures play an important role, since they ensure that the ranking criteria are 'sensitive to rare events' (Axiom 2). These measures define continuous linear real valued functions on L_∞, thus belonging to the dual space of L_∞ (Yosida, 1974), but cannot be represented by functions in L_1.

Example 7. A purely finitely additive measure that is not countably additive

The following defines a measure that is finitely additive but not countably additive and therefore cannot be represented by an integrable function in L_1: for every Borel set $A \subset R$, define $\rho(A)=1$ whenever $A \supseteq \{r : r > a, \text{for some} a \in R\}$, and $\rho(A)=0$ otherwise. ρ is a finitely additive set function but it is not countably additive, since R can be represented as a disjoint union of countably many bounded intervals, $R=\bigcup_{i=1}^{\infty}U_i$ each of which has zero measure $\rho(U_i)=0$, so the sum of their measures $\sum\rho(U_i)=0$ is zero, while by definition $\rho(R)=1$. This measure ρ cannot be represented by an L_1 function. A similar example can be provided for a purely finitely additive measure that is defined on bounded intervals of R.

References

Allais, M., 1988. The general theory of random choices in relation to the invariant cardinality function and the specific probability function. In: Munier, B.R. (Ed.), Risk, Decision and Rationality. Reidel, Dordrech, The Netherlands, pp. 233–289.
Brown, D., Lewis, 1981. Myopic economic agents. Econometrica, 49.
Chichilnisky, G., 1977. Non linear functional analysis and optimal economic growth. Journal of Mathematical Analysis and Applications 61, 504–520.
Chichilnisky, G., 1993. The cone condition, properness, and extremely desirable commodities. Economic Theory, 177–182.
Chichilnisky, G., 1996a. An axiomatic approach to sustainable development. Social Choice and Welfare 13, 257–321.
Chichilnisky, G., 1996b. Financial innovation in property Catastrophe reinsurance: the convergence of insurance and capital markets. Risk Financing Newsletter 403 13 (June (2)), 1–8.
Chichilnisky, G., 1998. Critical essays in mathematical economics volume III. In: The International Library of Critical Writings in Economics 93, Elgar Reference Collection, Cheltenham, UK/Northampton, MA, USA, p. 3.
Chichilnisky, G., 2000. An axiomatic approach to choice under uncertainty with Catastrophic risks. Resource and Energy Economics 22 (3), 221–231.
Chichilnisky, G., 2002. "Catastrophical Risk." Encyclopedia of Environmetrics, vol. 1. John Wiley & Sons, Ltd., Chicester, pp. 274–279.
Chichilnisky, G., 2006. "Catastrophic Risks: The Need for New Tools, Financial Instruments and Institutions", Privatization of Risk. Social Science Research Council, June 2006.
Arrow, K., 1971. Essays in the Theory of Risk Bearing. North Holland, Amsterdam.
Chichilnisky, G., Wu, H.-M., 2006. General equilibrium with endogenous uncertainty and default. Journal of Mathematical Economics 42 (May), 499–524.
Chichilnisky, G., Gorbachev, O., 2004. Volatility in the knowledge economy. Economic Theory 24 (September (3)), 531–547.
Chichilnisky, G., Gorbachev, O., 2005. In: Citanna, A., Donaldson, J., Polemarchakis, H., Siconolfi, P., Spear, S., (Eds.),"Volatility and Job Creation in the Knowledge Economy" (with O. Gorbachev) Essays in Dynamic General Equilibrium Theory Festschrift for David Cass. Series: Studies in Economic Theory, vol. 20. 45–74.
Chichilnisky, G., Heal, G.M., 1993."Competitive equilibrium in sobolev spaces without bounds on short sales", (with Geoffrey Heal), working paper no. 79, Institute for Mathematics and its Applications, University of Minnesota, June 1984. Journal of Economic Theory, 59 (April (2)), 364–384.
Chichilnisky, G., Heal, G.M., 1997. Social choice with infinite populations. Social Choice and Welfare 14 (2), 303–319.
Chichilnisky, G., Kalman, M., 1980. Application of functional analysis to models of efficient allocation of economic resources. Journal of Mathematical Analysis and Applications 30 (1), 19–32.
Chichilnisky, G., Shmatov, C., Avellaneda, M., 2005. Risk Measures in Growth and Value Investment", to Appear in Event Risk. Risk Books and Courant Institute for Mathematical Sciences, New York University, New York.
Chichilnisky, G., Kim, D., Wu, H.-M., 2005. "An Axiomatic Approach to Rare Events and Equity Premium" Working Paper, Columbia University, Yale University/National University of Taiwan.
Debreu, G., 1953. Valuation equilibrium and Pareto optimum. Proceedings of the National Academy of Sciences 40, 588–592.
De Groot, M.H., 2004. Optimal Statistical Decisions. John Wiley and Sons, Hoboken, NJ, 1970, and 2004.
Gilles, C., 1989. Charges as equilibrium prices and asset bubles. Journal of Mathematical Economics, 18.
Hernstein, N., Milnor, J., 1953. An axiomatic approach to measurable utility. Econometrica 21, 291–297.
Kahneman, M., Tversky, A., 1979. Prospect theory: an analysis of decisions under risk. Econometrica.
Le Doux, J., 1996. The Emotional Brain. Simon and Schuster.
Mas Collel, A., 1986. The equilibrium existence problem in topological vector lattices. Econometrica 54, 1039–1053.
Mehra, R., 2003. The equity premium: why is it a puzzle? Financial Analysts Journal, Jan/Feb 2003 pp. 54–69.
Majumdar, M., Radner, R., 1972. In: Balakrishnan, A.V. (Ed.), "Shadow Prices for Infinite Growth Paths", Techniques of Optimization. Academic Press, NY.
Malinvaud, E., 1953. Capital accumulation and efficient allocation of resources. Econometrica 21, 253–276.
McFadden, D., 1975. An example of the non existence of Malinvaud prices in a tight economy. Journal of Mathematical Economics 2, 17–19.
Mehra, R., Prescott, E.C., 1985. The equity premium: a puzzle. Journal of Monetary Economics 15 (March (2)), 145–161.
Radner, R., 1965. Maximal points of convex sets. In: Proceedings of the 5th Berkeley Symposium on Probability and Statistics. University of California Press, Berkeley, CA.
Tversky, A., Wakker, P., 1995. Risk attitudes and decision weights. Econometrica 6, 1225–1280.
Villegas, C., 1964. On quantitative probability σ- algebras. Annals of Mathematical Statistics 35, 1789–1800.
Weil, P., 1989. The equity premium puzzle and the risk-free rate puzzle. Journal of Monetary Economics 24 (November (3 I)), 401–421.
Yosida, K., 1974. Functional Analysis, 4th ed. Springer Verlag, New York/Heidelberg.
Yosida, K., Hewitt, E., 1952. Finitely level independent measures. Transactions of the American Mathematical Society 72, 46–66.

Please cite this article in press as: Chichilnisky, G., The topology of fear. J. Math. Econ. (2009), doi:10.1016/j.jmateco.2009.06.006