An Axiomatic Approach to Choice under Uncertainty with Catastrophic Risks

GRACIELA CHICHILNISKY*

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ABSTRACT. This paper analyses decision under uncertainty, motivated by problems emerging from global environmental risks. These are typically low probability events with major irreversible consequences. For such risks the Von Neumann Morgenstern Axioms for decision making under uncertainty are not appropriate, since they are insensitive to low probability events. The paper introduces an alternative set of axioms requiring sensitivity to both low and large probability events. Through a representation theorem in functional analysis, the results characterize all the operators whose maximization leads to the fulfillment of these axioms. They involve a convex combination of expected utility and a criterion based on the desire to avoid a low probability, and potentially catastrophic, events. It is shown that the new axioms help resolve the Allais paradox. Open questions about risk aversion, games under uncertainty and calculus of variations, are discussed.

1. INTRODUCTION

Global environmental risks such as climate change and rising sea levels are low probability events with widespread and possibly irreversible consequences. These are fundamentally new risks which are not well understood. Learning through experimentation is out of the question because these risks are effectively irreversible in a timescale that matters. As a result, classic theories which rely on expected utility may not work well because they underestimate low probability events, as discussed below. The need to make global environmental decisions calls for a systematic analysis of choices involving low probability events with major irreversible consequences. The topic is of current importance but has been neglected in the literature of choice under uncertainty.

This paper introduces a new decision making tool for such situations. Firstly, it shows why the classic Von Neumann axioms do not work well in this context, as they lead to expected utility that can be insensitive towards small probability events. Secondly, the paper introduces and develops a new set of axioms requiring sensitivity to both small and large probability events. These axioms appear to represent ways in which people rationalize the problem of making decisions in situations involving

*UNESCO Chair of Mathematics and Economics and Director, Program on Information and Resources, Columbia University, New York, N.Y. 10027. email:gc9@columbia.edu.

catastrophic risks. The axioms are different from the classic axioms by Von Neumann and Morgenstern, and lead to a different decision theory which is not based on expected utility analysis. Finally, through a representation theorem, I show that all the criteria implied by the new axioms have the following form: one term that takes into account the maximization of expected utility, plus a second term which is a well defined operator that can be interpreted as a desire to avoid a catastrophe. Both parts are present, and both turn out to be important in making decisions under catastrophic risks. The paper provides practical examples of how to use these criteria. It shows how the new axioms help explain the Allais paradox, which involves choices with low probability events, and suggests new questions on game theory and on the calculus of variations.

2. Von Neumann Morgenstern Axioms

A set of mathematical axioms introduced half a century ago by John Von Neumann and Oscar Morgenstern gave rise to a now classical tool for decision making under uncertainty. Several other mathematicians and economists, such as Hernstein, Milnor and Arrow, developed related axioms, [7]. The axioms formalize the properties of orders defined on sets of uncertain events; the orders are then used to rank or evaluate risky outcomes. The structure of the decision problem is simple. A system with uncertain characteristics is in one of several possible states; each state is the value of a random variable which describes the system. For example: the average temperature of the planet's surface is a state. The system's states can be described by real numbers.

To each state $s \in R$ there is an associated outcome, for example to each temperature level there is an associated vector describing soil fertility and precipitation. Therefore one has $x(s) \in R^N$, $N \geq 1$. A description of outcomes across all states is called a "lottery". A lottery is a function $x : R \rightarrow R^N$, and the space of all lotteries is therefore a function space L.

The Von Neumann-Morgenstern (NM) axioms provide a mathematical formalization of how to rank or order lotteries, i.e. of what are reasonable ways to order the elements of L. Optimization according to such an order defines decision making under uncertainty.

A main result obtained from the NM axioms is a representation theorem: a characterization of all the functionals on L which satisfy the NM axioms. Maximizing such a functional $W : L \rightarrow R$ over a constrained set given by initial conditions, defines rational choice under uncertainty. Von Neumann Morgenstern proved that an order over lotteries which satisfies their axioms admits a representation by an integral operator $W : L \rightarrow R$, which has as a kernel a countably additive measure over the set of states. Such operators are called "Von Neumann Morgenstern (NM) utilities" and the decision procedure obtained by optimizing such utilities is called "expected utility maximization", so that:

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s) \tag{1}$$

where the real line R is the state space, the function $x : R \to R^N$ is a "lottery", $u : R^N \to R$ is a "utility function" describing the utility provided by the outcome of the lottery in each state s, $u(s)$, and $d\mu(x)$ is a countably additive measure defining a probability distribution over measurable subsets of states in R. It is standard to require that the utility function is bounded to avoid the St. Petesburg paradox, see Chapter 3 of Arrow [2]. According to the NM representation theorem, rational choice under uncertainty which satisfies the NM axioms, must take the following form: a lottery x is ranked above another y if and only if W assigns to x a larger real number. In symbols:

$$x \succ y \Leftrightarrow W(x) > W(y),$$

where W satisfies (1).

The optimization of expected utility is a widely used procedure for evaluating choices under uncertainty. Mathematically, functionals such as W are convenient because they are amenable to a large body of knowledge which goes back several centuries: the calculus of variations. The Euler Lagrange equations are typically used to characterize optimal solutions. Such mathematical tools are widely used and very valuable to find and describe choices under uncertainty.

3. Catastrophic Risks

A catastrophic risk is a low probability event which can lead to major and typically irreversible losses. As already mentioned, global environmental problems have these characteristics. The classic methods defined above, despite their widespread use, are not satisfactory for evaluating catastrophic risks. The reasons are both practical and theoretical. From the practical point of view, it has been shown that using such criteria undervalues catastrophic risks and hence conflicts with the observed evidence of how humans evaluate such risks. For example using NM utilities, the most damaging scenarios of global climate change induce little if any economic loss. The Intergovernmental Panel on Climate Change (IPCC), the main international scientific organization in this area, recently predicted a highly contested figure of about 2% loss of economic value from a doubling of CO_2 concentration in the atmosphere. This is a symptom of a more general phenomenon: a simple computation shows that the hypothetical disappearance of all irrigation water in the USA and all the country's agricultural produce would have at most a 2 1/2 % impact on its Gross Domestic Product. This finding underscores the importance of using appropriate criteria for evaluating catastrophic risks.

Mathematically the problem arises from the fact that the expected utility operator W which emerges from the NM representation theorem (1) is defined with respect to a probability measure μ, which is therefore countably additive. Since the "utility"

function $u : R^N \rightarrow R$ is bounded (i.e.$\sup_{x \in R} \mid u(x) \mid < \infty$), the countable additivity of μ can be shown to imply that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$, where ε depends on x and y. To show this formally, introduce the following definition:

Definition 1. *A functional $W{:}L \rightarrow R$ is called "insensitive to small probability events" when*

$$W(x) > W(y) \Leftrightarrow \exists \varepsilon > 0 :$$

$$W(x') > W(y') \tag{2}$$

for all x', y' such that

$$x' = x \text{ and } y' = y \text{ a.e. on } A^c \subset R : \mu(A) < \varepsilon.$$

The interpretation of this definition is that W ranks x above y if and only if it ranks x' above y' for any pair of lotteries x' and y' which are obtained by modifying arbitrarily x and y in sets of states A with probability lower than ε. Under these conditions one says that the ranking defined by W is "insensitive" to the outcomes of the lottery in small probability events. The following lemma shows that, as defined by NM, the expected utility criterion W is not well suited for evaluating catastrophic risks. For simplicity of notation, and without loss of generality, let $N = 1$; the same results hold for arbitrary N.

Lemma 2. *Expected utility is insensitive to catastrophic risks.*

Proof. The expected utility criterion ranks lotteries in L as follows: $x(s) \succ y(s) \Leftrightarrow \exists$ a measurable and bounded utility function $u : R \rightarrow R$, and a probability measure μ on R :

$$\int_R u(x(s))d\mu(s) \succ \int_R u(y(s))d\mu(s).$$

Now

$$\int_R u(x(s))d\mu(s) \succ \int_R u(y(s))d\mu(s) \Leftrightarrow \exists \delta > 0 :$$

$$\int_R u(x(s))d\mu(s) \succ \int_R u(y(s))d\mu(s) + \delta.$$

Let

$$\epsilon = \epsilon(x, y) = \delta/6K$$

where

$$K = Sup_{x \in L, s \in R} \mid u(x(s) \mid .$$

If

$$x' = x \text{ and } y' = y \ a.e. \text{ on } S^c$$

where

$$\mu(S) < \epsilon,$$

then

$$| \int_R u(x(s))d\mu(s) - \int_R u(x'(s))d\mu(s) | < 2K\mu(S) < \delta/3,$$

and

$$| \int_R u(y(s))d\mu(s) - \int_R u(y'(s))d\mu(s) | < 2K\mu(S) < \delta/3.$$

Therefore

$$x \succ y \Rightarrow \int_R u(x'(s))d\mu(s) > \int_R u(y'(s))d\mu(s) \Rightarrow x' \succ y'.$$

Reciprocally:

$$x' \succ y' \Rightarrow x \succ y,$$

so that for $\epsilon = \delta/6K$

$$x \succ y \Leftrightarrow \exists \epsilon > 0 : x' \succ y' \text{ when } x = x' \text{ and } y = y' \ a.e. \text{ on any } S : \mu(S^c) < \epsilon$$

and therefore by definition the expectd utility criterion is insensitive to small probability events. ■

By the result just established, cost-benefit analysis under uncertainty based on expected utility maximization underestimates the outcomes of small probability events. It is biased against certain environmental projects which are design to prevent catastrophic events. Experimental evidence shows that humans treat choices under uncertainty somewhat differently from what the Von-Neumann Morgenstern axioms would predict, and raises questions about the need for alternative axioms which describe more accurately human's valuations.

4. Updating Von Neumann Morgenstern Axioms

Recently a new set of axioms has been developed which update Von Neumann Morgenstern Axioms to correct the bias pointed out in Section 3 against small probability events. Chichilnisky [4] introduced a well-defined set of axioms which contrast with Von-Neumann Morgenstern axioms, and produced the attendant representation theorems, identifying new types of functionals which are maximized under uncertainty. These axioms paralell similar ∀axioms and criterion for choice over time introduced in [5], [6].[1]

[1]See also Machina [8] for an alternative analysis to Von Neumann Morgenstern treatment of decision making under uncertainty. Machina does not provide an axiomatic treatment.

5. New Axioms for Choice Under Uncertainty

We propose three axioms for choice under uncertainty, which must be satisfied by the criterion $W : L \rightarrow R$ used to evaluate lotteries. The first axiom is satisfied by expected utility; the other two are not. The first axiom involves linearity and continuity of the criterion with respect to the utility derived from lotteries, where continuity is defined with respect to the sup norm on the space of utility values associated with lotteries L. Formally, utility values of lotteries L are in the space of measurable and essentially bounded functions on R , with the norm $\| u(x(s)) \| = Sup_{x \in L, s \in R} \mid u(x(s) \mid$.

- Axiom 1: continuity of the functional W with respect to its argument, the utility of the lottery $u(x)$[2]
- Axiom 2: Sensitivity to low probability events. This rules insensitivity to low probability events as in Definition 2 above
- Axiom 3: Sensitivity to large probability events. This rules out insensitivity to events of large probability, as defined below:

Definition 3. *A ranking is said to be insensitive to large probability events when* $\forall x, y \exists \epsilon > 0, \epsilon(x, y)$ *such that*

$$W(x) > W(y) \Leftrightarrow W(x') \geq W(y') \tag{3}$$

for all lotteries x', y' such that $x = x', y = y'$ a.e. on S^c where $\mu(S) > 1 - \epsilon$. In words: the ranking is the same on any two lotteries x' and y' that are obtained by modifying arbitrarily x and y in any bounded set of states $S \subset R$, which may have an arbitrarily large probability.

Example 4. *As a example of a function which is insensitive to large probability events, consider the space of all continuous linear real valued functions on* L_∞, *the "dual" of* L_∞, *denoted* L^*_∞. *Within this dual consider a "purely finitely additive measure"* ν *on* R *which assigns measure zero to any bounded set in* R, *i.e.* $\nu(S) = 0$ *if* $\forall x \in S, \mid x \mid < K$, *for some* $K > 0$. *Such measures define functionals satisfying (3). Such functionals are ruled out by axiom 3, which requires sensitivity to large probability events. Indeed, such functionals put all the "weight" on infinity, i.e. on events of arbitrarily small probabilities according to the countably additive measure* μ *on* R.

[2]Continuity is defined with respect to the sup norm on the space of utility values associated with lotteries L. The space of utility values of lotteries L is the space of all measurable essentially bounded functions on R , and the sup norm is defined as $\| u(x(s)) \| = Sup_{x \in L, s \in R} \mid u(x(s)) \mid$.

6. A Representation Theorem

Like the NM axioms, the three new axioms presented above lead to a representation theorem establishing the form of every ranking of lotteries that satisfies the three axioms given above. It has been shown in Chichilnisky 1992,[3] that there exist functionals $\Psi : L_\infty \to R$ which rank all lotteries and satisfy all the axioms. Rather than countably additive kernels, however, these functionals are a convex combination of integral operators with countably additive kernels and purely finitely additive measures, with both elements (countably and finitely additive) non-zero.

Theorem 5. *Any ranking $\succ$ of lotteries in $L = L_\infty(R)$ satisfying the three axioms defined must be of the form:*

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where $W : L \to R$

$$W(x) = \lambda[\int_R u(x(s))d\mu(s)] + (1-\lambda)\Phi(u(s(x)),$$

$$\begin{aligned} \text{for } \lambda &\in (0,1),\ u : R \to R,\ \mu \text{ a probability measure on } R, \\ \text{and } \Phi &: L \to R,\ \Phi \in L^* - L_1 \text{ is a 'purely finitely additive' measure.} \end{aligned}$$

Proof. The proof follows the line of argument presented in Chichilnisky ([6], [5]). As defined above, the space of all utility functions derived from lotteries is $L_\infty(R)$ with the sup norm.By axiom one, we are looking for an element of the dual space $L^*_\infty(R)$, the space of all continuous linear real valued functions on $L_\infty(R)$. By standard results in functional analysis, the dual space $L^*_\infty(R)$ consists of $L_1(R)$ as well as another space consisting of "purely finitely additive" measures, namely continuous linear functions that assign value zero to any function supported on a bounded set of R. By axioms two, the function W is not contained in L_1, since in that case as shown in Lemma 2, axiom two is violated. Axiom three implies that W is not a purely finitely additive measure either; as shown in Chichilnisky ([6], [5]) the only possible form is as represented above..

Example 6. *As an illustration of the representation theorem presented above, consider the case when the states are discrete, indexed by the integers Z. For each real number μ, $0 < \mu < 1$, a continuous linear functional $\Psi : l_\infty \to R$ can be defined as follows:*

$$\Psi(x) = \mu \sum_{s=1}^{\infty} \lambda^{-s} u(x(s)) + (1-\mu) \lim_{s \in Z} u(x(s)). \tag{4}$$

[3] Op. cit.

where $\lim_{s \in Z} u(x(s)$ is the (Hahn-Banach) extension of the continuous linear "limit" operator to the space l_∞ of all bounded real valued functions on Z. The interpretation of the two parts of the function Ψ in (4) is simple. The first part is an integral operator with an integrable kernel $\{\lambda^{-s}\}_{s \in Z}$ which defines a countably additive measure on Z and therefore emphasizes the weight of large probability events in the ranking of a lottery $x \in l_\infty$. The second part defines a purely finitely additive measure on Z which assigns positive weight to "small probability" events. It defines a measure with "heavy tails". Both parts are present, so Ψ is sensitive to small and large probability events. Catastrophic risks are therefore ranked more realistically by such functionals. The mathematics involved in these representation results is non linear analysis, as well as the analysis of convex systems.

7. Examples and Open Questions

7.1. Examples. Consider an electrical utility such as Con Edison in New York. They seek to implement a production and service plan which would be optimal under normal conditions, while at the same time avoiding a potentially catastrophic "black out" incident which could be costly in monetary terms and in human lives. Following our axioms and Theorem 5, a typical criterion that would be adopted would involve choosing among all possible plans so as to maximize the expected throughput plus minimizing the probability of reaching a critical level beyond which there would be a 'black out'. It can be shown that such a criterion would satisfy our three axioms.

7.2. The Allais Paradox. The first and perhaps most famous violation of the standard models of choice under uncertainty is due to M. Allais, who showed experimental evidence which is inconsistent with Von Neumann Morgenstern axioms. A variation of this paradox was reported by Kahneman and Tversky. They observed that 82% of the subjects chose a gamble A over another gamble B, and 83% of the subjects chose a gamble C over another D, so that at least 65% chose B and C. However, as shown below, this pair of gambles B and C is inconsistent with Von Neumann Morgenstern model of expected utility:

Example 7. *Gamble A consists of a .33chance of winning* \$2500, .66 *chance of winning* \$2400, *and* .01 *chance of winning* \$0, *while Gamble B is 1. chance of winning* \$2400. *Gamble C consists of* .33 *chance of winning* \$2500, *and* .67 *chance of winning* \$0, *while gamble D consists of* .34 *chance of winning* \$2400 *and* .66 *chance of winning* \$0.

Observe that if an individual prefers B over A, this means that their (sure) utility function u over income satisfies

$$u(2400) > .33u(2500) + .66u(2400) + .01u(0)$$